

Étale Groupoids, Derived Categories, and Operations

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Étale groupoids form an important class of topological groupoids, which includes (at least up to equivalence) the groupoids arising from foliations and from orbifolds, and groupoids associated to actions of discrete groups or infinitesimally free actions of Lie groups. Many constructions for spaces or manifolds extend to étale groupoids without much difficulty, while their extension to arbitrary groupoids becomes complicated or impossible.

The purpose of this paper is to describe how sheaf theory extends to étale groupoids. More specifically, we discuss the construction of the “derived category” of an étale groupoid, and show how the six operations of Grothendieck (namely, tensor, hom, f^* , f_* , $f_!$ and $f^!$) extend from spaces to étale groupoids. Thus, our purpose is similar to that of Bernstein and Lunts [BL], who discuss the derived category and the operations for groupoids associated to actions of compact Lie groups. However, it will become apparent that the case of étale groupoids is somewhat different in nature.

This paper makes no claim to originality and is purely expository, its only virtue perhaps being the presentation from the unifying viewpoint of derived categories. After an introductory section which provides the basic definitions and notation concerning étale groupoids, I describe the derived category of sheaves on a given étale groupoid. In section 3 the operations f^* and f_* , and the related cohomology of étale groupoids are discussed. These are well known, and in a sense simply a special case of that for sheaves on an arbitrary site [SGA IV]. The case of groupoids has been discussed in different degrees of generality and from different points of view in many sources, including [H2, H3, K, M2, M4]. There is an explicit discussion of Morita equivalence of groupoids (in section 4), and invariance of the operations under this type of equivalence. This invariance is of crucial importance, because in many cases the étale groupoid associated to a given geometric object is defined only up to Morita equivalence: this is the case, for example, for the holonomy groupoid of a foliation. In section 5 I discuss the operation $f_!$, which is related to compactly supported cohomology and homology of étale groupoids. This homology was first discussed in [CM1] (but without any reference to the derived category), and is closely related to the cyclic homology of the convolution algebra of an étale

Key words and phrases. étale groupoid, foliation, classifying space, derived category, cohomology.

2000 Mathematics Subject Classification. 58H05, 57R32.

Research partially supported by the Dutch Science Foundation (NWO).

groupoid [BN, C1]. The construction of $f_!$ — and of its right adjoint $f^!$ — is somewhat involved, partly because of complications arising from the fact that we cannot assume all étale groupoids to be Hausdorff. In fact, the main examples arising in the theory of foliations aren't Hausdorff. To overcome this difficulty, I have relied on [CM2]. In section 6 I give a combinatorial way of describing the derived category of an étale groupoid, which makes use of a small (discrete) category associated to the étale groupoid, called the embedding category. This category has the same cohomology and homology as the étale groupoid itself. Section 7 discusses the relation between the derived category of an étale groupoid and its classifying space.

Acknowledgements: I would like to thank the organisers of the conference at Boulder for inviting me, and for giving me the opportunity to contribute this paper to the proceedings. It is clear from the text (section 5) that I have made considerable use of joint work with Marius Crainic, all still to appear. I am indebted to many people for helpful discussions over the years about étale groupoids and related subjects. Besides Crainic, I would in particular like to mention W.T. van Est, A. Haefliger, and J. Mrcun. This research is partially supported by the Dutch Science Foundation (NWO).

1. Etale groupoids.

1.1. Groupoids (notation). A groupoid is a category in which every arrow is invertible [CWM]. Thus, a groupoid G is given by a set of objects G_0 and a set of arrows G_1 , together with structure maps satisfying the usual equations,

$$(1.1) \quad G_1 \times_{G_0} G_1 \xrightarrow{m} G_1 \xrightarrow[d_1]{d_0} G_0 \xrightarrow{u} G_1 \xrightarrow{i} G_1.$$

Here d_0 and d_1 are the source (domain) and target (codomain), u is the unit, i is the inverse, and m the multiplication defined on the fibered product $\{(h, g) \mid d_0(h) = d_1(g)\}$. We will write $g : x \rightarrow y$ to indicate that $g \in G_1$ is an arrow with $d_0(g) = x$ and $d_1(g) = y$. Furthermore, we write hg for $m(h, g)$, 1_x or id_x for $u(x)$ and g^{-1} for $i(g)$, all as usual.

1.2. Etale groupoids. A topological groupoid is a groupoid G , as above, in which G_0 and G_1 are each equipped with a topology for which all the structure maps in (1.1) are continuous. Such a groupoid is *étale* if $d_0 : G_1 \rightarrow G_0$ is a local homeomorphism (this implies that all other structure maps in (1.1) are local homeomorphisms as well). In many examples of étale groupoids, G_0 and G_1 are smooth manifolds and d_0 is a local diffeomorphism. One refers to G as a *smooth étale groupoid* in this case.

1.3. Examples.

- (a) Every topological space X can be viewed as an étale groupoid, with $X = G_0 = G_1$ and all structure maps in (1.1) identities.
- (b) If Γ is a groupoid acting continuously on a space X , the translation groupoid $\Gamma \ltimes X$ is the groupoid with X as space of objects and $\Gamma \ltimes X$ as space of arrows, an arrow $x \rightarrow y$ being a pair (γ, x) with $\gamma \cdot x = y$. This groupoid is étale if Γ is a discrete group.

- (c) Many examples of étale groupoids arise in the context of foliations. Haefliger's groupoid Γ^q which classifies codimension q foliations [H4] is an étale groupoid. If (M, \mathcal{F}) is a foliated manifold, its holonomy and monodromy groupoids aren't étale, but they become so if the space of objects is reduced to a complete transversal T . Different choices of transversals give Morita equivalent (§4) étale groupoids. Thus, to each foliation one can associate two étale groupoids, each well-defined up to Morita equivalence [H1, P].
- (d) Orbifolds give rise to étale groupoids [H1]. In fact, in the C^∞ -context orbifolds are essentially equivalent to étale groupoids which are proper, in the sense that $(d_0, d_1) : G_1 \rightarrow G_0 \times G_0$ is a proper map (see [MP]).

1.4. Homomorphisms. A (continuous) homomorphism $f : H \rightarrow G$ between étale groupoids consists of two continuous maps, $f_0 : H_0 \rightarrow G_0$ and $f_1 : H_1 \rightarrow G_1$, commuting with all the structure maps in (1.1) (e.g. $f_1(hk) = f_1(h)f_1(k)$, $d_0f_1(h) = f_0d_0(h)$, etc.). Thus, f is just a *continuous functor*. In a similar fashion, one can define continuous natural transformations τ , from one homomorphism $H \rightarrow G$ to another, as suitable maps $\tau : H_0 \rightarrow G_1$.

2. Sheaves.

In this section we introduce the category of G -sheaves for an étale groupoid G . We assume familiarity with the theory of sheaves on topological spaces, as exposed in, e.g., [B, Go, I]. Throughout this paper, the sheaves will be sheaves over a fixed field k (in practice \mathbb{R} or \mathbb{C}), and we will not explicitly refer to k in our notation. We remark, however, that in this section and the next it would have been sufficient to assume that k is a commutative ring, while in §4 this ring would have to be Noetherian and of finite cohomological dimension, exactly as for spaces [B].

2.1. The category $\text{Sh}(G)$. A G -sheaf (of k -modules) is a sheaf A on the space G_0 , equipped with a continuous right G -action. This action gives for each arrow $g : x \rightarrow y$ a (k -module) map

$$A_y \rightarrow A_x, \quad a \mapsto a \cdot g$$

between the stalks of A , and the usual conditions for an action ($(a \cdot g) \cdot h = a \cdot (gh)$ and $a \cdot 1 = a$) should be satisfied. Continuity of the action can most easily be expressed in terms of the map $A \times_{G_0} G_1 \rightarrow A$, $(a, y) \mapsto a \cdot g$, defined on the fibred product of $d_1 : G_1 \rightarrow G_0$ and the étale space $A \rightarrow G_0$ of the sheaf. A map between G -sheaves is a map $\varphi : A \rightarrow B$ of sheaves (of k -modules) on the space G_0 (in the usual sense), which in addition respects the action,

$$\varphi(a \cdot g) = \varphi(a) \cdot g.$$

In this way one obtains a well-defined category $\text{Sh}(G)$.

This category is the category of k -modules in the topos of G -sheaves of sets. In particular, it follows from the general properties of such categories that $\text{Sh}(G)$ is an abelian category with enough injectives.

There is an obvious functor, "forget the action", which we denote

$$(2.1) \quad F : \text{Sh}(G) \rightarrow \text{Sh}(G_0),$$

from $\text{Sh}(G)$ to the category of sheaves on the space G_0 of objects of the groupoid G . This functor is exact, and preserves injectives (the latter because G is étale).

2.2. Remark. For an étale groupoid G , there are always lots of G -sheaves. In fact, the sheaves on G_0 naturally constructed using the geometry of G_0 usually carry a G -action. For example, if G_0 is a smooth manifold, the sheaves Ω^q of differential q -forms on G_0 are G -sheaves. Many more examples of this kind can be found in the references at the end of this paper. (Notice, however, that this situation is completely different if G is not assumed étale. For instance, if G is a connected topological group, so that G_0 is a point, the functor F in (2.1) above is an equivalence of categories.)

For an étale groupoid G and an open set U in G_0 , one can define a sheaf $k[U]$ whose stalk at a point x is the free k -module on the set of arrows in G from x into some point of U . This sheaf is free on U , in the sense that morphisms of G -sheaves from $k[U]$ to a given G -sheaf A correspond to sections of A over U .

2.3. Complexes. We write $\text{Ch}(G)$, $\text{Ch}^+(G)$, $\text{Ch}^-(G)$ for the categories of cochain complexes (respectively bounded below and above such complexes) of G -sheaves of k -modules, and we use the obvious notions of G -equivariant chain maps and chain homotopies. Such a map $\varphi : A^* \rightarrow B^*$ is said to be a quasi-isomorphism (q.i.) if it induces isomorphisms between the cohomology G -sheaves. Since the functor F in (2.1) is exact, a map $A^* \rightarrow B^*$ is a q.i. between G -sheaves iff $F(A^*) \rightarrow F(B^*)$ is a q.i. of sheaves on the space G_0 in the usual sense.

2.4. Derived categories. The derived categories

$$\mathcal{D}(G), \mathcal{D}^+(G), \mathcal{D}^-(G)$$

of G -sheaves of k -modules are obtained, in the usual way, from the categories $\text{Ch}(G)$, $\text{Ch}^+(G)$ and $\text{Ch}^-(G)$, by inverting the q.i.'s. These categories can be described in a more concrete way as complexes of injective sheaves, and chain homotopy classes of maps, as usual.

2.5. Tensor and Hom. For two G -sheaves A and B , the tensor product sheaf $A \otimes B$ on G_0 carries a natural G -action, and defines the tensor product in the category $\text{Sh}(G)$. This tensor product extends to cochain complexes with the usual grading convention and, since we assume that k is a field, it induces a well-defined tensor product on derived categories.

Next, for G -sheaves A and B , the usual sheaf $\underline{\text{Hom}}(A, B)$ on G_0 (more precisely, this is the sheaf $\underline{\text{Hom}}(FA, FB)$) carries a natural G -action; to describe it explicitly, one uses again that G is étale. Thus $\underline{\text{Hom}}(A, B)$ is an object of $\text{Sh}(G)$. When A^* and B^* are cochain complexes of G -sheaves, one obtains a cochain complex, graded as usual, $\underline{\text{Hom}}(A, B)^n = \prod_{p-q=n} \underline{\text{Hom}}(A^p, B^q)$.

$\underline{\text{Hom}}$ and tensor are related by the usual adjunction formula, which is an isomorphism between G -sheaves,

$$(2.2) \quad \underline{\text{Hom}}(A \otimes B, C) = \underline{\text{Hom}}(A, \underline{\text{Hom}}(B, C)).$$

Applying $\underline{\text{Hom}}$ to injective resolutions gives a functor

$$\underline{\text{RHom}} : \mathcal{D}^-(G) \times \mathcal{D}^+(G) \rightarrow \mathcal{D}^+(G),$$

and the formula (2.2) holds when $=$ is interpreted as "canonical q.i.".

3. The operations f^* and f_* , and cohomology.

Let $f : H \rightarrow G$ be a homomorphism between étale groupoids (q.i.), fixed throughout this section.

3.1. The operation f^* . For a G -sheaf A , the pullback of A along $f_0 : H_0 \rightarrow G_0$ yields a sheaf $f^*(A)$ on H_0 with stalk $f^*(A)_y = A_{f(y)}$. This sheaf carries a natural action by the groupoid H . For an arrow $h : z \rightarrow y$ in H and $a \in f^*(A)_y$, this action is simply given in terms of the G -action on A by,

$$a \cdot h = a \cdot f(h).$$

In this way, we obtain a functor f^* making the following diagram commute (up to canonical isomorphism).

$$\begin{CD} \text{Sh}(G) @>f^*>> \text{Sh}(H) \\ @V F VV @VV F V \\ \text{Sh}(G_0) @>f_0^*>> \text{Sh}(H_0) \end{CD}$$

Here F is the forgetful functor, as before, and f_0^* denotes the usual pullback operation for sheaves on spaces. The functor f^* is exact (since f_0^* is), hence preserves q.i. between cochain complexes. So it induces an evident functor

$$f^* : \mathcal{D}^{(\pm)}(G) \rightarrow \mathcal{D}^{(\pm)}(H)$$

between derived categories.

3.2. The operation Rf_* . The functor f^* above has a right adjoint $f_* : \text{Sh}(H) \rightarrow \text{Sh}(G)$, uniquely determined by the isomorphism

$$\text{Hom}_H(f^*(A), B) = \text{Hom}_G(A, f_*(B))$$

for a G -sheaf A and an H -sheaf B . A more explicit description of $f_*(B)$ can be obtained from this isomorphism by taking for A a sheaf of the form $k[U]$ where $U \subseteq G_0$ is an open set (cf. 2.2), yielding

$$\Gamma(U, f_*(B)) = \text{Hom}_G(k[U], B),$$

where $\Gamma(U, -)$ denotes the k -module of sections over U . Unlike f^* , this functor f_* does not commute with the forgetful functor (i.e. $F \circ f_* \neq (f_0)_* \circ F$). However, f_* does preserve quasi-isomorphisms between injectives, hence induces a well-defined functor between derived categories

$$Rf_* : \mathcal{D}(H) \rightarrow \mathcal{D}(G),$$

adjoint to $f^* : \mathcal{D}(G) \rightarrow \mathcal{D}(H)$. The same holds for \mathcal{D}^+ (but not for \mathcal{D}^- since $Rf_* : \mathcal{D}(H) \rightarrow \mathcal{D}(G)$ doesn't map \mathcal{D}^- into \mathcal{D}^- , as is clear from 3.3 below).

3.3. Invariant sections and cohomology. Let A be a G -sheaf. A section $s \in \Gamma(G_0, A)$ is called invariant if, for any $g : x \rightarrow y$ in G ,

$$s(y) \cdot g = s(x).$$

Write $\Gamma_{\text{inv}}(G, A)$ for the k -module of invariant sections. The functor $\Gamma_{\text{inv}}(G, -)$ from G -sheaves to k -modules is a functor of the form f_* , where $f : G \rightarrow \text{pt}$ is the

unique homomorphism to a point. The cohomology of G with coefficients in A is defined in terms of Γ_{inv} , as

$$H^i(G, A) = R(\Gamma_{\text{inv}}(G, -)^i)(A) = H^i(\Gamma_{\text{inv}}(G, I^*))$$

where I^* is any injective resolution of A .

In the same way, one defines the hypercohomology $\mathbb{H}^i(G, A)$ for a cochain complex A .

This cohomology is the ordinary cohomology of the topos of G -sheaves, and has all the standard properties. If G is a space X (i.e. $G_0 = X = G_1$, see 1.3.(a)), it yields the usual sheaf cohomology of X , if G is a group it gives group cohomology and if G is an action groupoid it defines the standard equivariant sheaf cohomology [Gr].

4. Morita invariance.

In this section we explain that the category of G -sheaves only depends on the groupoid G up to Morita equivalence.

4.1. Essential equivalences. A homomorphism $f : H \rightarrow G$ between étale groupoids is called an *essential equivalence* if the following two conditions are satisfied.

(a) f is “fully faithful”, in the sense that

$$\begin{array}{ccc} H_1 & \xrightarrow{f_1} & G_1 \\ (d_0, d_1) \downarrow & & \downarrow (d_0, d_1) \\ H_0 \times H_0 & \xrightarrow{f_0 \times f_0} & G_0 \times G_0 \end{array}$$

is a fibered product.

(b) the map $d_1 \pi_1 : G_1 \times_{G_0} H_0 \rightarrow G_0$, sending a pair $(g : x \rightarrow x', y)$ with $f_0(y) = x$ to x' , is an open surjection.

4.2. Morita equivalence. Two étale groupoids G and G' are said to be Morita equivalent if there are two essential equivalences

$$G \leftarrow H \rightarrow G'$$

from a third groupoid H . This defines an equivalence relation.

The main property of Morita equivalence in the present context is the following (see [M1]).

4.3. Theorem. *If $f : H \rightarrow G$ is an essential equivalence then $f^* : \text{Sh}(G) \rightarrow \text{Sh}(H)$ is an equivalence of categories. Hence, if G and G' are Morita equivalent étale groupoids, then $\text{Sh}(G)$ and $\text{Sh}(G')$ are equivalent categories.*

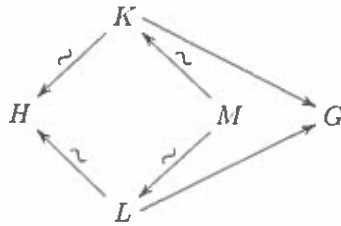
4.4. Remark. The converse is also true, at least for sheaves of sets: if G and G' define equivalent toposes of sheaves of sets, then G and G' are Morita equivalent (provided the spaces G_0 and G'_0 are ‘sober’ – this is a mild separation condition implied by Hausdorffness).

4.5. Corollary. *If $f : H \rightarrow G$ is an essential equivalence then $f^* : \mathcal{D}^{(\pm)}(G) \rightarrow \mathcal{D}^{(\pm)}(H)$ is an equivalence of categories. Hence, Morita equivalent étale groupoids have equivalent derived categories.*

4.6. Remark. It is possible to consider more general morphisms $f : H \rightarrow G$ between étale groupoids, given by equivalence classes of diagrams of homomorphism of the form,

$$(4.1) \quad H \xleftarrow{\sim} K \longrightarrow G,$$

where $K \xrightarrow{\sim} H$ is an essential equivalence. Two such diagrams $H \xleftarrow{\sim} K \rightarrow G$ and $H \xleftarrow{\sim} L \rightarrow G$ represent the same morphism iff there is a diagram of homomorphisms



commuting up to natural transformations, where arrows marked “ $\xrightarrow{\sim}$ ” are essential equivalences.

Two groupoids are Morita equivalent iff they become isomorphic in this category of general morphisms. This category can be described in various equivalent ways, e.g. in terms of principal bundles or in terms of topos morphisms, and plays a central role in the subject. See e.g. [HS, MRW, Pra, Pro, Mr].

A construction \mathcal{C} on groupoids is functorial on this general category iff it is functorial on ordinary homomorphisms and has the following two properties: first, if $\mathcal{C}(f) = \mathcal{C}(g)$ if there is a natural transformation between f and g ; and second, it turns essential equivalences into isomorphisms. (In fact, the first requirement is a consequence of the second.)

If \mathcal{C} takes categories as values, then it is functorial on general morphisms as soon as it is functorial on ordinary homomorphisms up to (coherent) natural isomorphism, and it turns essential equivalences into equivalences of categories (see [Pro]). For example, by this universal property, one can construct p^* and Rp_* for a generalised morphism $p : H \rightarrow G$. If p is represented by an essential equivalence $e : K \xrightarrow{\sim} H$ and a homomorphism $f : K \rightarrow G$, then $e^* : \mathcal{D}(H) \rightarrow \mathcal{D}(K)$ is an equivalence with inverse $e_* = Re_*$, and one defines $p^* = f^*e_*$ and $Rp_* = Rf_* \circ e^*$.

5. The operations $f_!$ and $f^!$.

The operations discussed so far — \otimes , Hom , f^* and Rf_* — are standard, and exist for any topos map, not just ones of the form $\text{Sh}(H) \rightarrow \text{Sh}(G)$ induced by a groupoid homomorphism. The construction of the operations $f_!$ and $f^!$ is more special, and we need some specific assumptions.

5.1. Overall assumptions. From now on we will assume that all topological spaces X have a basis of locally compact Hausdorff open sets $U \subseteq X$ of cohomological dimension bounded by a number d depending on X but not on U . In particular, if G is an étale groupoid, we will assume that G_0 and (hence) G_1 have this property. As the examples of holonomy groupoids of foliations show, it is not possible to assume that G_1 is Hausdorff (but one can always assume that G_0 is).

5.2. *c*-Soft sheaves. For a space X satisfying the conditions above, we call a sheaf A on X *c*-soft if for any Hausdorff open $U \subseteq X$, the restriction $A|_U$ is a *c*-soft sheaf on U (in the usual sense, [B]). Any sheaf has a finite *c*-soft resolution. The property of being *c*-soft is a local property. In particular, the pullback of a *c*-soft sheaf along an étale map is again *c*-soft. For an étale groupoid G , a G -sheaf is called *c*-soft if it is *c*-soft as a sheaf on G_0 .

5.3. Compact supports. There are some subtleties involved in extending the usual treatment of compact supports [B] to non-Hausdorff spaces. The reader will find a detailed discussion of this in [CM1]. Here we summarize the main points.

For such a *c*-soft sheaf, define $\Gamma_c(X, A)$ to be the subgroup of the group of all (set-theoretic, i.e. not necessarily continuous) sections of A , generated by sections σ which are continuous on a neighbourhood of their support, and have the property that this support is a compact subset of a Hausdorff open set $U \subseteq X$. This definition gives a functor

$$\Gamma_c : \text{Sh}(X) \rightarrow (k\text{-modules}),$$

extending the usual one for Hausdorff spaces, and still having all the standard properties.

5.4. $f_!$ for spaces ([CM1]). Let $f : X \rightarrow Y$ be a map between topological spaces. These spaces may be non-Hausdorff but satisfy the general assumptions of (5.1). There is an exact functor $f_!$ from sheaves on Y to sheaves on X , mapping *c*-soft sheaves to *c*-soft ones, and with the following properties, for any *c*-soft sheaf on Y , any $x \in X$ and any open $U \subseteq X$:

- (a) $\Gamma_c(U, f_!B) = \Gamma_c(f^{-1}(U), B)$.
- (b) $f_!(B)_x = \Gamma_c(f^{-1}(x), B)$.

Moreover, in a pullback square of the form

$$\begin{array}{ccc} Y' & \xrightarrow{q} & Y \\ f' \downarrow & & \downarrow f \\ X' & \xrightarrow{p} & X \end{array}$$

where p is étale (i.e. a local homeomorphism), the canonical map

$$f'_!q^*(B) \rightarrow p^*f_!(B)$$

is an isomorphism. (If p is not étale this property holds up to q.i., see [CM1]). Furthermore, if f is itself étale, and A is a *c*-soft sheaf on X , there is a natural morphism of sheaves

$$f_* : f_!f^*(A) \rightarrow A,$$

defined by summation along the fiber.

5.5. $f_!$ for groupoids. Now suppose $f : H \rightarrow G$ is a homomorphism between étale groupoids. Consider for each $n \geq 0$ the space $(G/f)_n = \bigcup_{x \in G_0} (G/f)_{n,x}$ of strings

$$(5.1) \quad (y_0 \xleftarrow{h_1} y_1 \leftarrow \dots \xleftarrow{h_n} y_n, f(y_n) \xleftarrow{g} x)$$

where h_1, \dots, h_n are arrows in H and g is one in G . Note that an arrow $u : x' \rightarrow x$ in G induces a map $(G/f)_{x',n} \rightarrow (G/f)_{x,n}$, sending the string (5.1) to (h_1, \dots, h_n, gu) .

In other words, the groupoid G acts on the space $(G/f)_n$. Write $\pi_n : (G/f)_n \rightarrow H_0$ for the map sending (5.1) to y_n

$$\pi_n(h_1, \dots, h_n, g) = \text{domain}(h_n),$$

and observe that π_n is a local homeomorphism since G and H are étale groupoids. Furthermore, define $\sigma_n : (G/f)_n \rightarrow G_0$ by

$$\sigma_n(h_1, \dots, h_n, g) = x = \text{domain}(g).$$

If B is a c -soft H -sheaf, then $\pi_n^*(B)$ is a c -soft sheaf on $(G/f)_n$, and one obtains a c -soft sheaf

$$f_!(B)_n := \sigma_{n!} \pi_n^*(B)$$

on G_0 . The action of G on $(G/f)_n$ induces an action of G on $f_!(B)_n$, so that $f_!(B)_n$ is a c -soft G -sheaf.

Next, the usual simplicial face maps

$$d_i : (G/f)_n \rightarrow (G/f)_{n-1} \quad (0 \leq i \leq n)$$

defined by $d_i(h_1, \dots, h_n, g) = (h_1, \dots, h_i, h_{i+1}, \dots, h_n, g)$ for $0 < i < n$, and by (h_2, \dots, h_n, g) for $i = 0$, and $(h_1, \dots, h_{n-1}, f(h_n)g)$ for $i = n$, make $(G/f)_n$ into a simplicial G -space. Notice that each face map d_i is étale. We observe that

$$\sigma_{n-1} \circ d_i = \sigma_n \quad (0 \leq i \leq n).$$

Furthermore,

$$\pi_{n-1} \circ d_i = \pi_n \quad (0 \leq i < n),$$

so that for an H -sheaf B as above there is a canonical isomorphism

$$d_i^* \pi_{n-1}^*(B) \cong \pi_n^*(B) \quad (0 \leq i < n).$$

For $i = n$, the action by H (of h_n on the stalk at (h_1, \dots, h_n, g)) gives an isomorphism

$$d_n^* \pi_{n-1}^*(B) \cong \pi_n^*(B)$$

for $i = n$ also. Thus, for each $i = 0, \dots, n$, summation along the fiber d_{i*} defines a map

$$d_{i!} \pi_n^*(B) \cong d_{i!} d_i^* \pi_{n-1}^*(B) \xrightarrow{d_{i*}} \pi_{n-1}^*(B).$$

Applying the functor $(\sigma_{n-1})_!$ to this map, one obtains a map of c -soft G -sheaves

$$\sigma_{n!} \pi_n^*(B) = (\sigma_{n-1})_! d_{i!} \pi_n^*(B) \xrightarrow{(\sigma_{n-1})_!(d_{i*})} (\sigma_{n-1})_! \pi_{n-1}^*(B),$$

or, in other words, a map

$$\delta_i : f_!(B)_n \rightarrow f_!(B)_{n-1} \quad (0 \leq i \leq n).$$

Thus $f_!(B)$ has the structure of a simplicial G -sheaf of k -modules. By taking alternating sums of the δ_i in the usual way, we obtain a chain complex of G -sheaves, still denoted $f_!(B)$.

Finally, if we start with a bounded above cochain complex of c -soft H -sheaves B^* , we obtain such a complex $f_!(B^*)$ by setting

$$f_!(B^*)^n = \bigoplus_{p-q=n} f_!(B^p)_q.$$

In [CM1] it is proved that this construction preserves quasi-isomorphisms, hence gives a well-defined functor

$$f_! : \mathcal{D}^-(H) \rightarrow \mathcal{D}^-(G).$$

(Here, it is convenient to represent objects of $\mathcal{D}^-(H)$ by bounded below complexes of c -soft sheaves. This is possible because each bounded below complex is q.i. to such a complex.)

5.6. Cohomology with Compact supports. For the special case of the unique morphism $p : G \rightarrow 1$ from any étale groupoid to the one-point groupoid, the construction above yields a functor $p_!$ from $\mathcal{D}^-(G)$ into bounded above cochain complexes of k -modules. We write

$$\mathbb{H}_c^i(G, B^*) = H^i(p_!(B^*)).$$

If B is (the c -soft resolution of) a single G -sheaf B concentrated in degree zero, this defines the cohomology of B with compact supports, denoted $H_c^i(G, B)$. (If G is a space, this is the usual cohomology with compact supports.)

We can now state the following version of "Verdier duality", essentially proved in [CM1].

5.7. Theorem. *Let $f : H \rightarrow G$ be a homomorphism between étale groupoids, as before. There exists a natural operation*

$$f^! : \mathcal{D}^+(G) \rightarrow \mathcal{D}^+(H),$$

uniquely determined by the property that there is a natural q.i.

$$(5.2) \quad \underline{\text{RHom}}(f_!B, A) \cong Rf_*(\underline{\text{RHom}}(B, f^!A))$$

for any two objects A of $\mathcal{D}^+(G)$ and B of $\mathcal{D}^-(H)$.

5.8. Remark. The isomorphism (5.2) is an isomorphism in $\mathcal{D}^+(G)$. Applying Rp_* for $p : G \rightarrow 1$ and taking cohomology in degree zero yields the adjunction formula

$$(5.3) \quad \text{Hom}_{\mathcal{D}^+(H)}(f_!B, A) = \text{Hom}_{\mathcal{D}^+(G)}(B, f^!A),$$

as usual. Conversely, it is possible to deduce the "internal" isomorphism (5.3) from the natural isomorphism (5.2) applied to localisations of G and H involved in the explicit description of Rf_* .

6. The embedding category of an étale groupoid.

The purpose of this section is to show how the derived category of an étale groupoid can be described purely combinatorially, in terms of a small category associated to the étale groupoid, first introduced in [M3] and called the embedding category there. First, we recall some basic constructions related to small categories.

6.1. Small categories. [CWM] These are 'discrete' categories, without a topology, and 'small' in the sense that their objects and arrows are sets rather than classes (relative to some ambient set theory). Let \mathcal{C} be such a category. A presheaf (of k -modules) on \mathcal{C} is a contravariant functor A from \mathcal{C} into k -modules. We write $A(C)$ for the value at an object $C \in \mathcal{C}$, and $\alpha^* : A(D) \rightarrow A(C)$ for the effect of an arrow $\alpha : C \rightarrow D$. A morphism between such presheaves is a natural transformation. This defines the category $\text{PSh}(\mathcal{C})$ of presheaves of k -modules on \mathcal{C} . With cochain complexes of presheaves defined in the obvious way, one defines a map $A^* \rightarrow B^*$ between such complexes to be a q.i. if $A^*(C) \rightarrow B^*(C)$ is a q.i. of cochain complexes of k -modules, for every object $C \in \mathcal{C}$. The associated derived categories are denoted

$$\mathcal{D}(\mathcal{C}), \mathcal{D}^+(\mathcal{C}), \mathcal{D}^-(\mathcal{C}).$$

The nerve $N(\mathcal{C})$ of \mathcal{C} is the simplicial set whose n -simplices are composable strings of the form $(C_0 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_n} C_n)$. The face maps d_i map such a string to $(C_0 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_i \alpha_{i+1}} \dots \xleftarrow{\alpha_n} C_n)$ for $0 < i < n$ and to $(C_1 \xleftarrow{\alpha_2} \dots \xleftarrow{\alpha_n} C_n)$ for $i = 0$ and $(C_0 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_{n-1}} C_{n-1})$ for $i = n$. The classifying space BC of \mathcal{C} is defined as the geometric realization of $N(\mathcal{C})$. The cohomology groups $H^n(\mathcal{C}, A)$ with coefficients in a presheaf A are constructed from the complex

$$C^n(\mathcal{C}, A) = \prod_{C_0 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_n} C_n} A(C_n),$$

where the product ranges over $N_n(\mathcal{C})$. For $f \in C^{n-1}(\mathcal{C}, A)$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in N_n(\mathcal{C})$, the coboundary is defined by

$$(df)(\alpha) = \sum_{i=0}^{n-1} (-1)^i f(d_i \alpha) + (-1)^n \alpha_n^* f(d_n \alpha).$$

If the presheaf A is constant, then this cohomology $H^*(\mathcal{C}, A)$ is isomorphic to the cohomology $H^*(BC, A)$ of the classifying space [Q].

6.2. The category $\text{Emb}(G)$. [M3] Let G be an étale groupoid. We construct a small category, called the embedding category of G and denoted $\text{Emb}(G)$. The construction depends on the choice of a basis of open sets for the topology on G_0 . Assume such a basis fixed from now on. (In practice there is often a natural choice for such a basis, e.g. consisting of (contractible) charts for a manifold structure on G_0 .) The objects of the category $\text{Emb}(G)$ will be these open sets in G_0 belonging to the chosen basis. An arrow $U \rightarrow V$ is given by a section $s : U \rightarrow G_1$ of the domain map $d_0 : G_1 \rightarrow G_0$, with the property that $d_1 \circ s$ defines an embedding $U \rightarrow V$. For emphasis, we denote by $\bar{s} : U \rightarrow V$ the arrow of $\text{Emb}(G)$ given by such a section s . One can think of \bar{s} as a U -parametrised family of arrows in the groupoid G , namely $\{s(x) : x \rightarrow d_1 s(x)\}_{x \in U}$. Composition of two arrows $\bar{s} : U \rightarrow V$ and $\bar{t} : V \rightarrow W$ in $\text{Emb}(G)$ is given by the formula

$$\bar{t} \circ \bar{s} = \overline{t * s}$$

where $(t * s)(x) = t(d_1 s(x)) \cdot s(x)$ (composition in G).

6.3. The functor Γ . Each G -sheaf A defines a presheaf $\Gamma(A)$ on $\text{Emb}(G)$, by

$$\Gamma(A)(U) = \Gamma(U, A),$$

the set of sections of A over U . If $\bar{s} : U \rightarrow V$ in $\text{Emb}(G)$ then $\bar{s}^* : \Gamma(V, A) \rightarrow \Gamma(U, A)$ is defined using the action of G , as $\bar{s}^*(a)(x) = a(d_1 s(x)) \cdot s(x)$. In this way one obtains a left exact functor

$$\Gamma : \text{Sh}(G) \rightarrow \text{PSh}(\text{Emb}(G)),$$

into the category of presheaves on the embedding category. Although not explicitly stated in this way, the arguments in [M3] show the following.

6.4. Theorem. *The functor Γ induces a full and faithful embedding of derived categories*

$$R\Gamma : \mathcal{D}^+(G) \rightarrow \mathcal{D}^+(\text{Emb}(G)).$$

(For the proof of this theorem, one uses that the functor Γ is itself full and faithful, and has a left adjoint Δ . For a presheaf B on $\text{Emb}(G)$, the G -sheaf $\Delta(B)$ has for its stalk at $x \in G_0$,

$$\Delta(B)_x = \lim_{\rightarrow x \in U} B(U),$$

where U ranges over all basic neighbourhoods of x . Then at the level of derived categories, Δ is left adjoint to $R\Gamma$, and the canonical map $\Delta(R\Gamma(A)) \rightarrow A$ is a q.i. for each cochain complex of presheaves A .)

6.5. Cohomology. For a G -sheaf A , the complex $R\Gamma(A)(U)$ computes the cohomology $H^*(U, A)$. In particular, if this cohomology vanishes in positive degrees for each open set U in the chosen basis, then the previous theorem implies that there is an isomorphism

$$H^*(G, A) = H^*(\text{Emb}(G), \Gamma(A)).$$

This is the case, for example, if A is (locally) constant while each open set U in the chosen basis is contractible. (In [M3], only this case is discussed.)

6.6. Remark. There are analogous statements concerning a functor $R\Gamma_c : \mathcal{D}^-(G) \rightarrow \mathcal{D}^-(\text{Emb}(G)^{\text{op}})$, and the relation between the compactly supported cohomology of G and the homology of the category $\text{Emb}(G)$. The category $\text{Emb}(G)$ is also a natural and convenient geometric object for the construction of characteristic classes [CM3, C2].

7. Relation to classifying spaces.

To conclude this paper, we briefly discuss the relation with the classifying space BG of an étale groupoid.

7.1. The classifying space BG . Let G be an étale groupoid. The nerve of G is the simplicial space $N(G)$ whose space $N_n(G)$ of n -simplices is the space of strings of arrows of the form

$$x_0 \xleftarrow{g_1} x_1 \longleftarrow \cdots \xleftarrow{g_n} x_n,$$

equipped with the fibered product topology,

$$N_n(G) = G_1 \times_{G_0} G_1 \times \cdots \times_{G_0} G_1.$$

The face maps $d_i : N_n(G) \rightarrow N_{n-1}(G)$ are defined as usual (see 6.1). The classifying space BG is the geometric realization of $N(G)$. Here we take the "thick" realization in the sense of [S].

7.2. Sheaves. Each G -sheaf A induces in a natural way a sheaf \tilde{A} on the space BG . This construction is explicitly described in [M4], where it is proved that there is a natural isomorphism in cohomology,

$$H^n(G, A) \xrightarrow{\sim} H^n(BG, \tilde{A}).$$

The same argument in fact shows the following somewhat stronger statement.

7.3. Theorem. *The construction $A \mapsto \tilde{A}$ defines a full embedding of derived categories*

$$\mathcal{D}^+(G) \hookrightarrow \mathcal{D}^+(BG).$$

7.4. Remark. The space BG is in general hard to deal with. It is an infinite dimensional space, and in the natural examples where G is not Hausdorff, neither is the space BG . The classifying space $B \text{Emb}(G)$ is better behaved in this respect — it is always a CW -complex. The main theorem in [M3] asserts that these two classifying spaces have the same weak homotopy type,

$$BG \cong B \text{Emb}(G).$$

However, I do not know whether Theorem 7.3 holds with $B \text{Emb}(G)$ instead of BG . I also do not know how to deal with compactly supported cohomology of G (and the related operations $f_!$ and $f^!$ in terms of BG).

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