Published in C. Albert et al (eds)
Integrable Systems and Foliations,
Birkhäuser, 1997

On the Weak Homotopy Type of Étale Groupoids

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To P. Molino, on the occasion of his 60th birthday

Etale groupoids play a central role in the theory of foliations. Well-known examples include the Haefliger groupoid Γ^q which classifies C^∞-foliations of codimension q [H71] and the holonomy groupoid of any foliation [W83]. In particular, invariants of leaf spaces of foliations are usually defined in terms of the classifying space or the C^*-algebra associated to this holonomy groupoid (see [C, H84, Mo, BN] and many others).

The purpose of this note is to prove that the classifying space of any étale groupoid G is weakly equivalent to that of a small (discrete) category, the arrows of which are continuous maps whose germs belong to G. (A precise formulation is given in Theorem 1.2 below.) This result can be viewed as a generalization of Segal's theorem [S78] on the weak homotopy equivalence between $B\Gamma^q$ and the classifying space of the monoid of smooth embeddings of $\mathbb{R}^q$ into itself (but the proof to be presented below is different from the one in [S78]).

1. Etale groupoids, and statement of the theorem

We begin by reviewing some standard definitions and fixing the notation. Recall that a groupoid is a small category in which every arrow has an inverse. A topological groupoid is a groupoid equipped with a topology that makes all the operations continuous. For a topological groupoid G, we use the notation G_0 for the space of objects of G, G_1 for the space of arrows, $s, t : G_1 \rightrightarrows G_0$ for the source and target maps, $m : G_1 \times_{G_0} G_1 \to G_1$, $m(f,g) = f \cdot g$ for the composition in G, $u : G_0 \to G_1$ for the units ($u(x) = 1_x$ is the identity arrow at x) and $i : G_1 \to G_1$ for the inverse ($i(x) = x^{-1}$). A point $g \in G_1$ with $s(g) = x$ and $t(g) = y$ is called an arrow from x to y in G, and denoted $g : x \to y$. The topological groupoid G is called *étale* if the source map $s : G_1 \to G_0$ is a local homeomorphism. (This implies that all other structure maps are local homeomorphisms also.)

If $g : x \to y$ is an arrow in an étale groupoid G, then one can choose a local section $\sigma_g : U_x \to G_1$ of the source map $s : G_1 \to G_0$ with $\sigma_g(x) = g$, to obtain a diffeomorphism $t \circ \sigma_g : U_x \to V_y$ into an open neighbourhood $V_y \subseteq G_0$. We denote by $\tilde{g}$ the germ of this diffeomorphism at x. The groupoid G is said to be *effective* if $\tilde{g} = \tilde{g}'$ implies $g = g'$ for any two parallel arrows $g, g' : x \to y$ in G. The notion of an effective étale groupoid is essentially equivalent to that of an S-atlas [vE].

For any topological groupoid G, one can construct its *classifying space* BG as the geometric realization of the simplicial space Nerve(G). This space BG represents the homotopy type of G.

Throughout this paper, we will work with the assumption that the space G_0 of objects has a basis of contractible open sets. This holds when G_0 is a manifold or a CW-complex, as is the case in all relevant examples.

1.1 Examples of étale groupoids. (a) Any topological space X gives rise to a trivial étale groupoid $\mathbf{X}$, with X as space of objects, and identity arrows only. The nerve of $\mathbf{X}$ is the constant simplicial space X, and $B\mathbf{X} = X$.

(b) If π is a group acting on a space X, one can form the *translation groupoid* (X, π). The objects of this groupoid are the points of X, and the arrows $x \to y$ are the elements $p \in \pi$ such that $p \cdot x = y$. The classifying space $B(X, \pi)$ is the familiar *Borel construction*, often denoted $E\pi \times_\pi X$ or X_π.

(c) [W83,H84] Let $(M, \mathcal{F})$ be a foliation of codimension q. The *holonomy groupoid* Hol$(M, \mathcal{F})$ has as space of objects the disjoint union $U = \coprod U_i$ of a family of transversally embedded q-dimensional disks, big enough so that each leaf is met. An arrow $x \to y$ in the holonomy groupoid is an equivalence class of paths $\alpha : [0,1] \to M$ from x to y which stay in one single leaf; two such paths α and α' are equivalent, i.e. represent the same arrow, if the holonomy of the loop $\alpha^{-1} \cdot \alpha'$ at x is trivial. The classifying space $BHol(M, \mathcal{F})$ serves as a model for the *leaf space* of $(M, \mathcal{F})$; its (weak) homotopy type does not depend on the choice of U.

(d) [H71, S78, B] The *Haefliger groupoid* Γ^q has $\mathbb{R}^q$ as space of objects; arrows $x \to y$ are germs at x of diffeomorphisms from an open neighbourhood of x to an open neighbourhood of y. The topology on the space of arrows is the sheaf topology, making Γ^q into an étale groupoid.

(e) *Orbifolds* can be modelled by étale groupoids [H84]; in [MP] it is proved that they correspond exactly to étale groupoids G with the property that $(s, t) : G_1 \to G_0 \times G_0$ is a proper map.

(f) Effective étale groupoids are well-known to be basically equivalent to pseudogroups (e.g. [Mo, p. 2, 266]). If Γ is a pseudogroup of transformations of a manifold M, then the germs of elements of Γ form an effective étale groupoid with M as space of objects. Conversely, from an étale groupoid G, one can construct a pseudogroup of transformations of the space G_0, in a way similar to the construction of the category Emb(G), below.

We now associate to each étale groupoid G a small (discrete) category Emb(G) of "G-embeddings". This construction depends on the choice of a basis of contractible open sets for the space G_0. We assume that such a choice has been made, and suppress it from the notation. The objects of the category Emb(G) are the basic open sets $U \subseteq G_0$. An arrow $U \xrightarrow{\sigma} V$ in Emb(G) is a section $\sigma : U \to G_1$ of the source map $s : G_1 \to G_0$ such that $t \circ \sigma$ is an embedding of U into V. These arrows can be composed in the obvious way, by using the composition of the given groupoid G. Explicitly, if $U \xrightarrow{\sigma} V$ and $V \xrightarrow{\tau} W$ are arrows, then $\tau \circ \sigma : U \to W$ is the section $U \to G_1$ defined by $(\tau \circ \sigma)(x) = \tau(t\sigma x) \cdot \sigma(x)$ (where the dot on the right is the composition in G). Note that each arrow $\sigma : U \to V$ in Emb(G) induces an actual imbedding $\bar{\sigma} = t \circ \sigma : U \hookrightarrow V$; but two different arrows may induce the same such embedding.

1.2 Theorem. *For any étale groupoid G, the classifying space BG has the same weak homotopy type as the classifying space of the category* Emb(G).

1.3 Examples. (a) Let X be a topological space with a given basis of contractible open sets, and $\mathbf{X}$ the associated trivial groupoid as in 1.1 (a). In this case, Emb($\mathbf{X}$) is the category (poset) of basic open sets in X and inclusions. For this special case, Theorem 1.2 states that X has the same weak homotopy type as the classifying space BEmb($\mathbf{X}$); this is the geometric realization of the simplicial set with as typical n-cell a string of inclusions $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_n$ between basic open sets.

(b) Consider the translation groupoid (X, π) associated to a group action as in 1.1(b). For this groupoid, the theorem states that the Borel construction $E\pi \times_\pi X$ has the same weak homotopy type as BEmb(X, π); the latter is the realization of the simplicial set with as typical n-cell a string

$$U_0 \xleftarrow{p_0} U_1 \longleftarrow \cdots \xleftarrow{p_n} U_n,$$

where $U_i \subseteq X$ are basic open sets, $p_i \in \pi$, and $p_i \cdot U_i \subseteq U_{i-1}$.

(c) Let $G \subseteq \Gamma^q$ be any étale groupoid which contains all germs of affine transformations. Let B be the basis for the open sets of $\mathbb{R}^q$ consisting of all disks $B(x, \varepsilon) = \{y \in \mathbb{R}^q | d(x, y) < \varepsilon\}$. Fix a disk $B = B(O, 1)$ around the origin. In the category Emb(G), any disk $B(x, \varepsilon)$ is isomorphic to B. Thus Emb(G) is categorically equivalent [CWM, p. 91] to the full subcategory with B as its unique object. This is the monoid of all arrows $B \to B$ in Emb(G), which we denote by Emb$_B$(G). Then the theorem specializes to the following:

1.4 Corollary. *For any étale groupoid $G \subseteq \Gamma^q$ as above, BG has the same weak homotopy type as the classifying space of the discrete monoid* Emb$_B$(G).

For $G = \Gamma^q$, one recovers as a special case Segal's theorem [S78] which states that $B\Gamma^q$ is weakly homotopy equivalent to BM, where M is the monoid of all smooth embeddings of $\mathbb{R}^q$ into itself.

2. Sheaf cohomology

In this section, we describe a sheaf cohomology for étale groupoids. This cohomology has been introduced in [H76] and discussed further in [H92]; it can also be viewed as the cohomology of a topos [SGA4, vol.2], viz. that of the topos BG of G-equivariant sheaves of sets which is discussed in some detail in [M91]. The definitions apply equally well to an arbitrary topological category (see [S68] for this notion). For topological categories, we will use the same notation $G = (G_0, G_1, s, t, m, u)$ as for topological groupoids above, except that there is no operation i for inverse, of course. Such a topological category G will be called *étale* (or more precisely, *s-étale*) if its source map, $s : G_1 \to G_0$, is a local homeomorphism. Thus, any étale groupoid provides an example of an étale topological category. Another source of examples comes from diagrams of spaces:

2.1 Example. A *diagram of spaces* indexed by a small (discrete) category I is a covariant functor Y from I into the category of topological spaces. From such a diagram, one can construct a topological category, often denoted $\int_I Y$, whose objects are pairs (i, y) where i is an object of I and $y \in Y(i)$. An arrow $\alpha : (i, y) \to (j, z)$ in this category is an arrow $\alpha : i \to j$ in I with the property that $Y(\alpha)(y) = z$. The topology on this category $\int_I Y$ is that of the disjoint sum: the space of objects is topologized as $\sum_{i \in I} Y(i)$, and the space of arrows can be identified with the sum $\sum_{\alpha : i \to j} Y(i)$, where α ranges over all arrows in I. Observe that $\int_I Y$ is s-étale. Also note that there is an obvious continuous projection functor $\pi : \int_I Y \to I$ into the discrete category I.

From now on, G denotes a fixed étale groupoid, or an étale topological category.

A *G-sheaf* is a sheaf A of abelian groups on the space G_0 of objects, equipped with an action by G, which we write from the right. Thus, each arrow $g : x \to y$ in G induces a group homomorphism $A_y \to A_x$, $a \mapsto a \cdot g$, between the stalks. This action is required to be continuous, as a map $A \times_{G_0} G_1 \to A$.

With the obvious notion of action preserving homomorphism, these G-sheaves form a abelian category with enough injectives.

For a G-sheaf A, a global section $\sigma : G_0 \to A$ is said to be G-invariant if $\sigma(y) \cdot g = \sigma(x)$ for any arrow $g : x \to y$ in G. The group of all such global sections is denoted

$$\Gamma_{inv}(G, A).$$

The cohomology of G with coefficients in A is defined as the right derived functor of $\Gamma_{inv}(G, -)$. Thus, for any resolution $0 \to A \to I^0 \to I^1 \to \cdots$ of A by injective G-sheaves, one has by definition

$$H^n(G, A) = H^n(\Gamma_{inv}(G, I^\cdot)).$$

2.2 Examples. Referring to the examples in 1.2, one recovers in (a) the ordinary sheaf cohomology of a space X, and in (b) the equivariant sheaf cohomology. When G_0 is a point, so that G is just a (discrete) group, one recovers the usual Eilenberg-Mac Lane group cohomology. When G is a small discrete category, one recovers the usual cohomology of small categories. Note that in all these special cases, Theorem 2.3 below is well-known.

Now let L be a locally constant abelian sheaf on the classifying space BG (i.e., a twisted system of coefficients on BG). By pulling back along the obvious map $G_0 \hookrightarrow BG$, one obtains a sheaf $\hat{L}$ on G_0. The arrows of G are realized as arcs in BG, and by path-lifting in L one obtains a well-defined continuous G-action on $\hat{L}$. Thus $\hat{L}$ has the structure of a G-sheaf. We will use the following result:

2.3 Theorem. [M95] *For any étale topological category G and any twisted system of coefficients L on BG, there is a natural isomorphism*

$$H^n(BG, L) \cong H^n(G, \hat{L}) \quad (n \geq 0).$$

3. Proof of Theorem 1.2

In this section, G denotes a fixed étale groupoid, as in the statement of the theorem. For such a G, we have constructed the associated discrete category $\mathrm{Emb}(G)$ of G-embeddings. To compare the classifying spaces BG and $B\mathrm{Emb}(G)$, we will introduce an auxiliary topological category $\mathcal{E}(G)$.

There is an obvious "inclusion" functor Y from $\mathrm{Emb}(G)$ into the category of topological spaces, and $\mathcal{E}(G)$ is defined as the associated s-étale topological category $\int_{\mathrm{Emb}(G)} Y$, described in 2.1. Thus, an object of $\mathcal{E}(G)$ is a pair (U, x) where U is an object of $\mathrm{Emb}(G)$ (i.e. a basic open set in G_0) and $x \in U$; an arrow $(U, x) \xrightarrow{\sigma} (V, y)$ is a G-embedding $U \xrightarrow{\sigma} V$ with the property that the associated map $\bar{\sigma} : U \to V$ sends x to y. There are continuous functors

$$\mathrm{Emb}(G) \xleftarrow{\pi} \mathcal{E}(G) \xrightarrow{\lambda} G;$$

the functor π is the obvious projection functor defined on objects by $\pi(U, x) = U$ (cf. 2.1), while λ is defined on objects by $\lambda(U, x) = x$; furthermore, λ sends an arrow $\sigma : (U, x) \to (V, y)$ in $\mathcal{E}(G)$ to the arrow $\sigma(x) : x \to y$ in G. These two functors π and λ induce mappings between classifying spaces

$$B\mathrm{Emb}(G) \xleftarrow{B\pi} B\mathcal{E}(G) \xrightarrow{B\lambda} BG,$$

and we will prove that each of these mappings is a weak homotopy equivalence. For π, this is easy; it is a special case of the following lemma.

3.1 Lemma. *Let Y be a diagram of spaces indexed by a small category I, as in 2.1. If each of the spaces $Y(i)$ in the diagram is contractible, then the projection $\pi : \int_I Y \to I$ induces a weak homotopy equivalence $B\int_I Y \to BI$.*

Proof. The functor π induces a map of simplicial spaces $\mathrm{Nerve}(\int_I Y) \to I$. In degree n, this map can be written as

$$\sum_{i_0 \leftarrow \cdots \leftarrow i_n} Y(i_n) \to \sum_{i_0 \leftarrow \cdots \leftarrow i_n} pt,$$

(where pt is the one-point space). Since each space $Y(i_n)$ is assumed contractible, this map in degree n is a (weak) homotopy equivalence. The lemma follows by the well-known fact [see e.g. [BK, S74] that a map between simplicial spaces which is a weak homotopy equivalence in each degree also induces a weak homotopy equivalence between the geometric realizations.

The other half of the theorem is given by the following proposition.

3.2 Proposition. *The functor $\lambda : \mathcal{E}G \to G$ induces a weak homotopy equivalence $B\lambda : B\mathcal{E}(G) \to BG$.*

For the proof of this proposition, we will use the Whitehead theorem, and prove that the map $B\lambda : B\mathcal{E}(G) \to BG$ induces isomorphisms in cohomology with twisted coefficients (Lemma 3.4) as well as an isomorphism of fundamental groups (Lemma 3.5). For the isomorphism in cohomology, we will use the description in terms of sheaves provided by Theorem 2.3. To this end, observe that the continuous functor $\lambda : \mathcal{E}(G) \to G$ induces by pullback an evident functor

$$\lambda^* : (G\text{-}sheaves) \to (\mathcal{E}(G)\text{-}sheaves). \tag{1}$$

This functor is exact, and has a right adjoint λ_* (this means that there is a natural isomorphism $\mathrm{Hom}_G(A, \lambda_* B) \cong \mathrm{Hom}_{\mathcal{E}(G)}(\lambda^* A, B)$). These functors λ^* and λ_* exist for any continuous functor λ between topological categories. The following lemma, however, is special to the functor λ under consideration.

3.3 Lemma. *There exists a functor $\lambda_! : (\mathcal{E}(G)\text{-sheaves}) \to (G\text{-sheaves})$ which is left exact, and left adjoint to the functor λ^* in (1); i.e., there is a natural isomorphism*

$$\mathrm{Hom}_G(\lambda_!(B), A) \cong \mathrm{Hom}_{\mathcal{E}(G)}(B, \lambda^* A), \tag{2}$$

for any G-sheaf A and any $\mathcal{E}(G)$-sheaf B.

Proof. An $\mathcal{E}(G)$-sheaf B can be described as follows: For each object U of $\mathrm{Emb}(G)$ (i.e., each basic open set $U \subseteq G_0$) one has a sheaf B_U on the space U. Furthermore, for each arrow $\sigma : U \to V$ in $\mathrm{Emb}(G)$, with associated map $\tilde{\sigma} : U \to V$, one has a map

$$B(\sigma) : \tilde{\sigma}^*(B_V) \to B_U \tag{3}$$

of sheaves on U. These maps should satisfy the usual functoriality conditions. Thus, for a composition $U \xrightarrow{\sigma} V \xrightarrow{\tau} W$, the diagram

$$\begin{array}{ccc} \tilde{\sigma}^*\tilde{\tau}^*(B_W) & \xrightarrow{\sim} & (\widetilde{\tau\sigma})^*(B_W) \\ {\scriptstyle \tilde{\sigma}^* B(\tau)} \downarrow & & \downarrow {\scriptstyle B(\tau\sigma)} \\ \tilde{\sigma}^*(B_V) & \longrightarrow & B_U \end{array}$$

is required to commute. From such a B we will construct a G-sheaf $\lambda_!(B)$.

To begin with, we define a presheaf $p(B)$ on the space G_0 of objects of G. For a basic open set $U \subseteq G_0$, define

$$p(B)(U) = \Gamma(U, B_U).$$

For an inclusion $U \subseteq V$, the unit section $u : U \to G_1$ (where $u(x) = 1_x$ is the identity at x in G) defines an arrow $u : U \to V$ in $\mathrm{Emb}(G)$, for which the associated actual map $\tilde{u} : U \to V$ is simply the inclusion. The structure map (of the form (3)) $B(u) : B_V|U \to B_U$ thus gives rise to a homomorphism

$$\Gamma(B, B_V) \xrightarrow{\text{res}} \Gamma(U, B_V|U) \xrightarrow{\Gamma(U, B(u))} \Gamma(U, B_U), \tag{4}$$

and this is by definition the restriction map $p(B)(V) \to p(B)(U)$ for the presheaf $p(B)$.

Let $\lambda_!(B)$ be the associated sheaf of $p(B)$. Thus $\lambda_!(B)$ is the sheaf on G_0 with as stalk at a point x the direct limit

$$\lambda_!(B)_x = \varinjlim_{x \in U} \Gamma(U, B_U), \tag{5}$$

where U runs over all basic open neighbourhoods of x. This sheaf $\lambda_!(B)$ has the structure of a G-sheaf, with the action by arrows of G described as follows. For an arrow $g : x \to y$ in G, we need to define a homomorphism "action by g",

$$(-) \cdot g : \lambda_!(B)_y \to \lambda_!(B)_x. \tag{6}$$

To do this, choose a section $\sigma : W \to G_1$ of the source map $s : G_1 \to G_0$, defined on a neighbourhood W of x, with $\sigma(x) = y$. For an element $b \in \lambda_!(B)_y$, choose a basic neighbourhood V of y so small that b is represented by an element $b \in \Gamma(V, B_V)$ (cf. (5)). Next, let $U \subseteq W$ be a basic neighbourhood of x, so small that $t \circ \sigma(U) \subseteq V$. Then σ defines an arrow $U \to V$ in $\mathrm{Emb}(G)$, and hence induces a homomorphism $B(\sigma) : \bar{\sigma}^*(B_V) \to B_U$ of sheaves. We define $b \cdot g \in \lambda_!(B)_x$ to be the element represented by the image of $b \in \Gamma(V, B_V)$ under the composition

$$\Gamma(V, B_V) \xrightarrow{\bar{\sigma}^*} \Gamma(U, \bar{\sigma}^*(B_V)) \xrightarrow{\Gamma(U, B(\sigma))} \Gamma(U, B_U).$$

This defines the required action map (6). It is clear from the explicit construction of $b \cdot g$ that this action-map is continuous in b and g. Thus $\lambda_!(B)$ is a G-sheaf.

From the description (5) of the stalks of $\lambda_!(B)$, it is clear that $\lambda_!$ is a left-exact functor. Thus, to prove the lemma, it remains to verify the adjunction isomorphism (2) in the statement of the lemma. For an $\mathcal{E}(G)$-sheaf B as above, and any G-sheaf A, this isomorphism is given by constructing sheaf maps

$$\lambda_!(B) \xrightarrow{\varphi} A \quad \text{and} \quad B \xrightarrow{\psi} \lambda^*(A)$$

from each-other, as follows. Suppose given a map ψ; such a map consists of a system of maps $\psi_U : B_U \to \lambda^*(A)_U = A|U$ of sheaves on U, one for each basic open $U \subseteq G_0$. Taking sections, one obtains maps $\Gamma(U, B_U) \to \Gamma(U, A)$, which together give a map of presheaves $p(B) \to A$ on G_0, and hence a map $\lambda_!(B) \to A$ for the associated sheaves. It is readily verified that this map respects the given G-action on A and the G-action on $\lambda_!(B)$ just constructed.

Conversely, given a map $\varphi : \lambda_!(B) \to A$, one can construct a map $\psi : B \to \lambda^*(A)$ as follows. For each basic open U we need to define a map $\psi_U : B_U \to A|U$ of sheaves on U. Let $x \in U$ and let $b \in B_{U,x}$ be a point in the stalk over x. To define $\psi_U(b) \in A_x$, choose a section $\xi : N_x \to B$ through b, defined on a neighbourhood $N_x \subseteq U$ of x. From the arrow $u : N_x \to U$ in $\mathcal{E}(G)$ given by the unit section $u : N_x \to G_1$, the $\mathcal{E}(G)$-sheaf structure of B gives a map

$$\Gamma(U, B(u)) : \Gamma(N_x, B_U) \to \Gamma(N_x, B_{N_x})$$

(exactly as in (4)). Define

$$\psi_U(b) = \Gamma(U, B(u))(\xi)(x).$$

This completes the description of the map $\psi : B \to \lambda^*(A)$ from a given map $\varphi : \lambda_!(B) \to A$. We omit the straightforward verification that ψ is a well-defined map of $\mathcal{E}(G)$-sheaves, and that these two constructions, of ψ from φ and conversely, are mutually inverse.

3.4 Lemma. *For any abelian G-sheaf A, the functor λ induces an isomorphism in cohomology*

$$\lambda^* : H^n(G, A) \xrightarrow{\sim} H^n(\mathcal{E}(G), \lambda^*(A)) \quad (n \geq 0).$$

Proof. By 3.3, the functor λ^* in (1) preserves injective resolutions. Thus the lemma follows immediately from the fact that λ^* induces an isomorphism of invariant sections $\Gamma_{inv}(G, A) \xrightarrow{\sim} \Gamma_{inv}(\mathcal{E}(G), \lambda^*(A))$. □

Observe that 3.4 applies to all sheaves, and not just to locally constant ones as required for Proposition 3.2.

As explained above, the following lemma completes the proof of Proposition 3.2, and hence that of Theorem 1.2.

3.5 Lemma. *The functor $\lambda : \mathcal{E}(G) \to G$ induces for each base-point x an isomorphism $\pi_1(B\mathcal{E}(G), x) \to \pi_1(BG, \lambda(x))$.*

Proof. It suffices to prove that the category of covering spaces of BG is equivalent to that of the category of covering spaces of $B\mathcal{E}(G)$. Now, by the construction $L \mapsto \tilde{L}$ preceding the statement of Theorem 2.3, applied to sheaves of sets, the first category is equivalent to the category of locally constant G-sheaves – that is, the category of G-equivariant covering spaces of G_0. Similarly, the second category, of covering spaces of $B\mathcal{E}(G)$, is equivalent to the category of locally constant $\mathcal{E}(G)$-sheaves. This is the category of those $\mathcal{E}(G)$-sheaves B such that, using the notation of the proof of 3.1, the two conditions

(i) each sheaf B_U on U is constant, and

(ii) each map $B(\sigma) : \sigma^*(B_V) \to B_U$ is an isomorphism,

hold. It thus suffices to show that the functor λ^* of (1) above restricts to an equivalence on these categories of locally constant sheaves. For this, observe first that the functors λ^* and $\lambda_!$ on sheaves both preserve the property of being locally constant. Furthermore, for any G-sheaf A, the counit $\lambda_!\lambda^*A \to A$ of the adjunction of Lemma 3.3 is an isomorphism, because the functor λ^* is full and faithful (cf. [CWM, p. 88]). Finally, from the explicit conditions (i) and (ii), one easily sees that the unit $B \to \lambda^*\lambda_!B$ is an isomorphism whenever B is locally constant. This proves the lemma. □

Acknowledgements

This research is partially supported by The Netherlands Organisation for Scientific Research (NWO). I would like to thank A. Haefliger for some helpful comments on an earlier version (July 1995) of this paper.

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Received: October, 1995.