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Bisimplicial Sets and the Group-Completion Theorem

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This note makes no claim to originality. Its aim is to give a short, conceptual proof of the so-called group-completion theorem. It was written for these proceedings at the suggestion of J. F. Jardine, and it owes a lot to discussions he and I had at the Chateau. I am also indebted to A. Joyal for some helpful suggestions.

Let M be a topological monoid and BM its classifying space. The group-completion theorem asserts that the homomorphism of Pontryagin rings $H_*(M) \to H_*(\Omega BM)$ (integral homology) induced by the canonical map $M \to \Omega BM$ is the universal solution to inverting the multiplicative subset $\pi_0(M) \subset H_*(M)$, i.e. $H_*(M)[\pi_0(M)^{-1}] \xrightarrow{\sim} H_*(\Omega BM)$, provided $\pi_0(M)$ lies inside the center of $H_*(M)$. Several proofs of this result or variants thereof have been given in the literature, see e.g. [1], [7], [12], [8], [5], and it plays an important rôle in K-theory.

The present proof really comes down to some elementary category-theoretic considerations (in particular, it does not use spectral sequences or quasi-fibrations). It makes use of a certain closed model structure on bisimplicial sets, that I will discuss first. I wish to point out, however, that the only thing needed from §1 is the factorization constructed in the proof of CM5 below.

§1. Bisimplicial sets

Let $\mathcal{E}$ be the category of simplicial sets, $\mathcal{E} = \text{Sets}^{\Delta^{\text{op}}}$, so that $\mathcal{E}^{\Delta^{\text{op}}}$ is the category of bisimplicial sets. I write

$$\gamma^* : \mathcal{E} \to \mathcal{E}^{\Delta^{\text{op}}}$$

for the functor sending a simplicial set E to the corresponding constant functor $\Delta^{\text{op}} \to \mathcal{E}$ with value E. A more interesting functor is the *diagonal*

$$\delta^* : \mathcal{E}^{\Delta^{\text{op}}} \to \mathcal{E};$$

writing $\mathcal{E}^{\Delta^{\text{op}}} = \text{Sets}^{(\Delta\times\Delta)^{\text{op}}}$, δ^* is given by composition with the diagonal $\Delta \to \Delta \times \Delta$. It is an elementary fact from category theory (see [6]) that δ^* has both a left adjoint $\delta_!$ and a right adjoint δ_*:

$$\mathcal{E} \underset{\delta_*}{\overset{\delta_!}{\underset{\longleftarrow}{\overset{\longrightarrow}{\xrightarrow{\;\delta^*\;}}}}} \mathcal{E}^{\Delta^{\text{op}}}$$

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These adjoints are constructed by so-called Kan extension. In particular, $\delta_!$ is completely described by the fact that it commutes with colimits, together with the formula

$$\delta_!(\Delta[n]) = \Delta[n,n],$$

where $\Delta[n] \in \mathcal{E}$ and $\Delta[n,n] \in \mathcal{E}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ are the obvious representable functors, as usual (i.e., as a functor $(\boldsymbol{\Delta} \times \boldsymbol{\Delta})^{\mathrm{op}} \longrightarrow$ Sets, $\Delta[n,n] = \mathrm{Hom}_{(\boldsymbol{\Delta}\times\boldsymbol{\Delta})}(-,([n],[n]))$). A basic fact concerning δ^* is the following (cf. [3]).

1.1 Lemma. *If $E \longrightarrow E'$ is a map in $\mathcal{E}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ such that $E_n \longrightarrow E'_n$ is a weak equivalence in $\mathcal{E}$ for each n, then $\delta^*(E) \longrightarrow \delta^*(E')$ is also a weak equivalence.*

Recall that a weak equivalence of simplicial sets is a map whose geometric realization induces isomorphisms in homotopy groups. If one moreover defines fibrations in $\mathcal{E}$ to be Kan fibrations, and cofibrations in $\mathcal{E}$ to be monomorphisms, then this gives $\mathcal{E}$ the structure of a closed model category, see [9], [10]. Recall that a trivial (co)fibration is a (co)fibration which is also a weak equivalence.

Now define fibrations, cofibrations, and weak equivalences (w.e.'s) in $\mathcal{E}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ as follows: $E \xrightarrow{f} E'$ is a fibration (resp. a w.e.) in $\mathcal{E}^{\boldsymbol{\Delta}^{\mathrm{op}}}$ if and only if $\delta^*(f) : \delta^*(E) \longrightarrow \delta^*(E')$ is a fibration (resp. a w.e.) in $\mathcal{E}$, and $E \xrightarrow{f} E'$ is a cofibration if and only if f has the left lifting property (LLP, see [9], p. I.5.1) with respect to all trivial fibrations in $\mathcal{E}^{\boldsymbol{\Delta}^{\mathrm{op}}}$.

1.2 Proposition. *This defines a closed model structure on $\mathcal{E}^{\boldsymbol{\Delta}^{\mathrm{op}}}$, and $\delta^* : \mathcal{E}^{\boldsymbol{\Delta}^{\mathrm{op}}} \longrightarrow \mathcal{E}$ induces an equivalence of the associated homotopy categories* $\mathrm{Ho}(\mathcal{E}^{\boldsymbol{\Delta}^{\mathrm{op}}}) \xrightarrow{\sim} \mathrm{Ho}(\mathcal{E})$.

As a preparation for the proof, consider the following two bisimplicial sets:

$$A^k[n] = \delta_!(\Lambda^k[n]) = \bigcup_{j\neq k}\{\Delta[n-1,n-1] \overset{(\hat{j},\hat{j})}{\hookrightarrow} \Delta[n,n]\}$$
$$\dot{\Delta}[n,n] = \delta_!(\dot{\Delta}[n]) = \bigcup_{0\leq j\leq n}\{\Delta[n-1,n-1] \overset{(\hat{j},\hat{j})}{\hookrightarrow} \Delta[n,n]\}.$$

1.3 Lemma. *$A^k[n] \hookrightarrow \Delta[n,n]$ is a trivial cofibration in $\mathcal{E}^{\boldsymbol{\Delta}^{\mathrm{op}}}$, and $\dot{\Delta}[n,n] \hookrightarrow \Delta[n,n]$ is a cofibration in $\mathcal{E}^{\boldsymbol{\Delta}^{\mathrm{op}}}$.*

Proof: Since $\delta_!$ is left adjoint to δ^*, it is clear that both inclusions are cofibrations. So we only have to show that $\delta^*\delta_!$ sends $\Lambda^k[n] \hookrightarrow \Delta[n]$ to a weak equivalence. Of course, it is enough to show that $|\delta^* A^k[n]|$ is contractible. Write

$$A^k[n]_{\ell,m} = \{([\ell] \xrightarrow{\alpha} [n], [m] \xrightarrow{\beta} [n]) \mid \exists j \neq k \,:\, \alpha,\beta \text{ both miss } k\},$$

and consider the projection

$$A^k[n] \xrightarrow{\pi_1} \gamma^*\left(\Lambda^k[n]\right). \tag{1}$$

For a fixed ℓ, π_1 is the map of simplicial sets

$$\coprod_{[\ell]\xrightarrow{\alpha}[n]} \Bigg(\bigcup_{\substack{j\neq k \\ j\notin \mathrm{im}(\alpha)}} F^j[n] \Bigg) \longrightarrow \coprod_{\substack{[\ell]\xrightarrow{\alpha}[n] \\ \exists j\neq k : j\notin \mathrm{im}(\alpha)}} \Delta[0]$$

where $F^j[n] \subset \Delta[n]$ is the j-th face. Consider a particular $\alpha : [\ell] \to [n]$. If $\mathrm{im}(\alpha) \cup \{k\} = [n]$, the corresponding summands on both sides of (2) are empty. If $\mathrm{im}(\alpha) \cup \{k\} \neq [n]$, the corresponding summand on the left-hand side is a non-empty union of faces of $\Delta[n]$ which all have the vertex k in common, hence is contractible. So by 1.1, δ^* sends (1) to a weak equivalence, and therefore $|\delta^* A^k[n]|$ is contractible. This proves the lemma.

PROOF OF 1.2: I use the version CM1–CM5 of the axioms from [**10**, p. 233], which I will remind you of in the course of the proof. CM1 asserts the existence of finite limits and colimits, CM2 asserts that if two out of $f, g, f \circ g$ are weak equivalences then so is the third, and CM3 is the axiom that a retract of a fibration (respectively, cofibration or weak equivalence) is again one. These three axioms are obviously satisfied.

CM5 states that each map can be factored in two ways: as a trivial cofibration followed by a fibration, and as a cofibration followed by a trivial fibration. But since $\delta_!$ is left adjoint to δ^*, a map f in $\mathcal{E}^{\Delta^{\mathrm{op}}}$ is a fibration (respectively a trivial fibration) if and only if f has the right lifting property with respect to all inclusions $A^k[n] \hookrightarrow \Delta[n,n]$ (respectively, $\dot{\Delta}[n,n] \hookrightarrow \Delta[n,n]$). So by 1.3, the usual"small object argument" ([9], [4]) proves CM5. Since this is the only thing really relevant for the sequel, let me give the details for the case of the first factorization. Let $X \xrightarrow{f} Y$ be any map in $\mathcal{E}^{\Delta^{\mathrm{op}}}$. Consider all commutative squares of the form

$$\begin{array}{ccc} A^k[n] & \xrightarrow{\alpha} & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \Delta[n,n] & \xrightarrow{\beta} & Y \end{array}$$

and index them by indices i (calling the corresponding maps $A^{k_i}[n_i] \xrightarrow{\alpha_i} X$, $\Delta[n_i, n_i] \xrightarrow{\beta_i} Y$). Now form the pushout

$$\begin{array}{ccc} \coprod_i A^{k_i}[n_i] & \xrightarrow{\{a_i\}} & X \\ \downarrow & & \downarrow{\scriptstyle u_i} \\ \coprod_i \Delta[n_i, n_i] & \xrightarrow{\{\beta_i\}} & X_1 \end{array}$$

and let $X_1 \xrightarrow{p_1} Y$ be the unique factorization given by $X \xrightarrow{f} Y$ and $\coprod_i \Delta[n_i, n_i] \xrightarrow{\{\beta_i\}} Y$. Repeat this with f replaced by p_1, to get $p_1 = p_2 u_2$, then with f replaced by p_2, etc. This gives a sequence

$$X = X_0 \overset{u_1}{\hookrightarrow} X_1 \overset{u_2}{\hookrightarrow} X_2 \hookrightarrow \dots$$

and maps $X_i \xrightarrow{p_i} Y$ with $p_i u_i = p_{i-1}$ (where $p_0 = f$). Let $Z = \bigcup_n X_n$, and let $X \overset{u}{\hookrightarrow} Z$ be the corresponding inclusion. The p_i together give a map $Z \xrightarrow{p} Y$ with $pu = f$. Clearly p is a fibration, because it has the RLP with respect to all inclusions $A^k[n] \hookrightarrow \Delta[n,n]$; and u is not only a trivial cofibration, but has the LLP with respect to any fibration whatsoever.

Finally, we prove CM4, which says that for a commutative square

$$\begin{array}{ccc} A & \longrightarrow & E \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & X \end{array}$$

in $\mathcal{E}^{\Delta^{op}}$ where u is a cofibration and p a fibration, a diagonal lifting $B \to E$ exists if either u or p is a weak equivalence.

If p is a trivial fibration, a lifting exists by definition of the cofibrations. If u is a trivial cofibration, we first factor u as $q \circ j$ where q is a fibration and j has the LLP with respect to all fibrations (by the proof of CM5 just given). Then q is a trivial fibration, so if we construct successive liftings

$$\begin{array}{ccccc} A & & \longrightarrow & & E \\ {\scriptstyle j}\downarrow & & \overset{h}{\nearrow} & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{q} & B & \longrightarrow & X \end{array}$$

and

$$\begin{array}{ccc} A & \xrightarrow{j} & C \\ {\scriptstyle u}\downarrow & \overset{i}{\nearrow} & \downarrow{\scriptstyle q} \\ B & = & B \end{array}$$

then $h \circ i$ is the required lifting.

Let us now prove that $\mathcal{E} \xrightarrow{\delta^*} \mathcal{E}^{\Delta^{op}}$ induces an equivalence of model categories. Let $\mathcal{K} \subset \mathcal{E}$ be the full subcategory of Kan complexes, and let $\mathcal{F} \subset \mathcal{E}^{\Delta^{op}}$ be the full subcategory of fibrant objects (i.e., objects for which the map into the terminal object is a fibration). Clearly, δ^* maps $\mathcal{F}$ into $\mathcal{K}$. Moreover, since $\delta^*(A^k[n] \hookrightarrow \Delta[n,n])$ is a trivial cofibration in $\mathcal{E}$ by 1.2, it follows from the adjunctions between $\delta_!$, δ^* and δ_* that δ_* maps $\mathcal{K}$ into $\mathcal{F}$:

$$\begin{array}{ccc} \mathcal{E} & \underset{\delta_*}{\overset{\delta_*}{\rightleftarrows}} & \mathcal{E}^{\Delta^{\mathrm{op}}} \\ \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} \\ \mathcal{K} & \underset{\delta^*}{\overset{\delta_*}{\rightleftarrows}} & \mathcal{F} \end{array}$$

It is thus enough to show that (i) for $X \in \mathcal{K}$, the counit $\delta^*\delta_*(X) \xrightarrow{\varepsilon} X$ is a w.e., and that (ii) for $F \in \mathcal{F}$, the unit $E \xrightarrow{\eta} \delta_*\delta^*(E)$ is a w.e. (ii) follows from (i) by definition of the w.e.'s in $\mathcal{E}^{\Delta^{\mathrm{op}}}$. For (i), consider the components of the counit $\delta^*\delta_*(X) \xrightarrow{\varepsilon} X$,

$$\varepsilon_n : \delta^*\delta_*(X) = \mathrm{Hom}(\Delta[n] \times \Delta[n], X) \longrightarrow \mathrm{Hom}(\Delta[n], X) = X_n, \quad \varepsilon_n(f) = f \circ d$$

where $\Delta[n] \xrightarrow{d} \Delta[n] \times \Delta[n]$ is the diagonal. Let $p : \Delta[n] \times \Delta[n] \longrightarrow \Delta[n]$ be the first projection, and let $\pi : X \longrightarrow \delta^*\delta_*(x)$ have components $\pi_n : \mathrm{Hom}(\Delta[n], X) \longrightarrow \mathrm{Hom}(\Delta[n] \times \Delta[n], X)$ given by $\pi_n(g) = g \circ p$. Then $\varepsilon \circ \pi = \mathrm{id}_X$, so it is enough to show that π is a weak equivalence. Let $\xi : \gamma^*(X) \longrightarrow \delta_*(X)$ be defined by components $\xi_{n,m} : \mathrm{Hom}(\Delta[n], X) \longrightarrow \mathrm{Hom}(\Delta[n] \times \Delta[m], X)$, $\xi_{n,m} = g \circ p$, where $p : \Delta[n] \times \Delta[m] \longrightarrow \Delta[n]$ is the projection. Then $\pi = \delta^*(\xi)$. Moreover, for fixed m, $\xi_{-,m} : X \hookrightarrow X^{\Delta[m]}$ is the canonical inclusion which is a weak equivalence if X is Kan. So π is a w.e. by 1.1, and therefore so is ε. This proves 1.2.

§2. Simplicial categories acting on simplicial sets

Let h_* be some homology or homotopy theory (defined on $\mathcal{E}$). An h_*-equivalence is a map inducing isomorphisms in h_*. We will need that the pushout of an inclusion which is an h_*-equivalence is again one, and that h_* commutes with filtered colimits. (Then the analogue of 1.1 holds for h_*-equivalences.) The main examples to keep in mind are $h_* = H_*$ (integral homology), or $h_* = \pi_*$ (homotopy), but h_* can also be a generalized homology theory (cf. [2], appendix).

Let $\mathbb{C}$ be a category object in $\mathcal{E}$, given by domain and codomain maps of simplicial sets $\mathrm{mor}(\mathbb{C}) \underset{d_1}{\overset{d_0}{\rightrightarrows}} \mathrm{ob}(\mathbb{C})$, etc, and let X be a $\mathbb{C}$-diagram in $\mathcal{E}$. So X is given by maps $X \xrightarrow{\pi} \mathrm{ob}(\mathbb{C})$ (the projection) and $\mathrm{mor}(\mathbb{C}) \times_{\mathrm{ob}(\mathbb{C})} X \longrightarrow X$ (the action) satisfying the usual identities. I write $X_{\mathbb{C}}$ for the category of elements (also called the translation category): $X_{\mathbb{C}}$ is the simplicial category whose space of objects is X, whose space of morphisms is $\mathrm{mor}(\mathbb{C}) \times_{\mathrm{ob}(\mathbb{C})} X$ (pullback along d_0 =domain), and whose domain and codomain maps are the projection $\mathrm{mor}(\mathbb{C}) \times_{\mathrm{ob}(\mathbb{C})} X \xrightarrow{\pi_2} X$ and the action $\mathrm{mor}(\mathbb{C}) \times_{\mathrm{ob}(\mathbb{C})} X \xrightarrow{\cdot} X$ respectively; composition in $X_{\mathbb{C}}$ comes from composition in $\mathbb{C}$. The projection map $X \xrightarrow{\pi} \mathrm{ob}(\mathbb{C})$ gives an obvious functor $X_{\mathbb{C}} \xrightarrow{\pi} \mathbb{C}$ of simplicial categories, and therefore by taking the nerve, a map $N(X_{\mathbb{C}}) \xrightarrow{N(\pi)} N(\mathbb{C})$ in $\mathcal{E}^{\Delta^{\mathrm{op}}}$. If $C \in \mathrm{ob}(\mathbb{C})_0$

is a vertex of ob($\mathbb{C}$), we can form the pullback (1) in $\mathcal{E}^{\Delta^{op}}$.

(1)
$$\begin{array}{ccc} \gamma^*(X(C)) & \longrightarrow & N(X_{\mathbb{C}}) \\ \downarrow & & \downarrow{\scriptstyle N(\pi)} \\ 1 & \xrightarrow{C} & N(\mathbb{C}) \end{array}$$

2.1 THEOREM. *Assume $\mathbb{C}$ has a discrete space of objects. If for each vertex $\alpha \in mor(\mathbb{C})_0$, the map $X(d_0\alpha) \to X(d_1\alpha)$ given by the action of $\mathbb{C}$ on X is an h_*-equivalence in $\mathcal{E}$, then $X(C)$ is h_*-equivalent to the homotopy fiber of $\delta^*(N(\pi))$.*

(Recall that if $E \xrightarrow{p} B$ is a map in $\mathcal{E}$ and $1 \xrightarrow{b} B$ is a vertex of B, the homotopy fiber of p is the pullback $Y \times_B E$, where $1 \to Y \to B$ is a factorization of $1 \xrightarrow{b} B$ into a trivial cofibration followed by a fibration.)

PROOF: Factor $1 \xrightarrow{C} N(\mathbb{C})$ as $1 \xrightarrow{j} Y \xrightarrow{q} N(\mathbb{C})$ where j is a trivial cofibration and q is a fibration in $\mathcal{E}^{\Delta^{op}}$, as in the proof of CM5 above. So j is a colimit of a sequence of pushouts of coproducts of maps of the form $A^k[n] \xhookrightarrow{u} \Delta[n,n]$. Since pullbacks along $N(\pi)$ commute with colimits (of objects over $N(\mathbb{C})$), it is enough to show that for each composite

(2)
$$A^k[n] \xhookrightarrow{u} \Delta[n,n] \xrightarrow{\sigma} N(\mathbb{C})$$

δ^* sends the map obtained by pullback along $N(\pi)$,

$$u^*\sigma^*(X_{\mathbb{C}}) \to \sigma^*(N(X_{\mathbb{C}}))$$

to an h_*-equivalence. A map $\sigma : \Delta[n,n] \to N(\mathbb{C})$ corresponds to an element $C_0 \xrightarrow{\sigma_1} C_1 \longrightarrow \cdots \xrightarrow{\sigma_n} C_n$ in $N(\mathbb{C}_n)$, i.e., σ_i are morphisms in $\mathbb{C}_n$, but C_i are objects in ob($\mathbb{C}$)$_0$ (since ob($\mathbb{C}$) is discrete by assumption). There is a map in $\mathcal{E}^{\Delta^{op}}$

(4)
$$\mu : \Delta[n,n] \times \gamma^*(X(C_0)) \to \sigma^*(N(X_{\mathbb{C}}))$$

induced by the action of $\mathbb{C}$ on X. Explicitly,

$$\sigma^*(N(X_{\mathbb{C}}))_{k,\ell} = \{(\alpha,\beta,x) \mid [k] \xrightarrow{\alpha} [n],\ [\ell] \xrightarrow{\beta} [n],\ x \in X(C_{\alpha(0)})_\ell\},$$

while

$$(\Delta[n,n] \times \gamma^*(X(C_0)))_{k,\ell} = \{(\alpha,\beta,x) \mid [k] \xrightarrow{\alpha} [n],\ [\ell] \xrightarrow{\beta} [n],\ x \in X(C_0)_\ell\},$$

and

$$\mu(\alpha,\beta,x)=\beta^*(\sigma_{\alpha(0)}\circ\cdots\circ\sigma_1)\,.\,x\in X(C_{\alpha(0)})_\ell.$$

For a fixed k,

$$\mu_k\,:\,\coprod_{[k]\xrightarrow{\alpha}[n]}\Delta[n]\times X(C_0)\longrightarrow\coprod_{[k]\xrightarrow{\alpha}[n]}\Delta[n]\times X(C_{\alpha(0)})$$

is a coproduct of maps $\Delta[n]\times X(C_0)\to\Delta[n]\times X(C_{\alpha(0)})$ which have h_*-equivalences $X(C_0)\to X(C_{\alpha(0)})$ (action by $\Delta[0]\longrightarrow\Delta[n]\xrightarrow{\sigma_{\alpha(1)}\circ\cdots\circ\sigma_1}\text{mor}(\mathbb{C})$) as deformation retracts, so μ_k is an h_*-equivalence. By the h_*-analogue of 1.1, $\delta^*(\mu)$ is an h_*- equivalence. Similarly, the action of $\mathbb{C}$ on X induces a map

$$\mu'\,:A^k[n]\times\gamma^*(X(C_0))\to u^*\sigma^*N(X_{\mathbb{C}})$$

such that $\delta^*(\mu')$ is an h_*-equivalence. But we have a commutative diagram

$$\begin{array}{ccc} A^k[n]\times\gamma^*X(C_0) & \xrightarrow{u\times\text{id}} & \Delta[n,n]\times\gamma^*X(C_0)\\ {\scriptstyle\mu'}\downarrow & & \downarrow{\scriptstyle\mu}\\ u^*\sigma^*N(X_{\mathbb{C}}) & \longrightarrow & \sigma^*N(X_{\mathbb{C}}) \end{array}$$

where $\delta^*(\mu')$ and $\delta^*(\mu)$ are h_*-equivalences as we have just seen, and $\delta^*(u\times\ \text{id})=\delta^*(u)\times\ \text{id}$ is an h_*-equivalence by 1.3 (the domain and codomain of $\delta^*(u)$ are contractible). So the lower horizontal map must be an h_*-equivalence as well, as was to be shown.

2.2 Remark: If in 2.1 $\mathbb{C}$ is discrete (i.e., $\mathbb{C}$ is a category in Sets rather than in $\mathcal{E}$) then the case where $h_*=\pi_*$ is essentially equivalent to Quillen's theorem B (see **[11]**), as is well-known.

§3. The group-completion theorem

From 2.1, the group-completion theorem follows easily, by an argument given in **[8]**. For the convenience of the reader, I give a somewhat modified version of their argument here.

3.1 Corollary. *Let M be a topological monoid. Then the canonical map of M-spaces $M\to\Omega BM$ induces an isomorphism $H_*(M)[\pi_0(M)^{-1}]\xrightarrow{\sim}H_*(\Omega BM)$ (as asserted in the beginning of this note), provided $\pi_0(M)$ is contained in the center of $H_*(M)$.*

Before proving this corollary, let us note tht if S is a countable multiplicative subset contained in the center of a ring R, and A is a (right) R-module, then the universal R-module $A[S^{-1}]$ (with the property that multiplication by any $s\in S$ is a bijection) can be constructed as the colimit of the sequence

$$A\xrightarrow{\rho_{s_1}}A\xrightarrow{\rho_{s_2}}A\to\ \dots$$

where ρ_{s_i} is (right) multiplication by s_i, and $(s_i : i \in \mathbb{N})$ is an enumeration of S in which each element occurs infinitely often.

PROOF OF 3.1: Of course, we may equivalently prove the case of a simplicial monoid (= a simplicial category with 1 as its space of objects). Now first notice that both $H_*(M)[\pi_0(M)^{-1}]$ and $H_*(\Omega BM)$ are functors of M which commute with filtered limits. Moreover, if $\pi_0 M$ is in the center of $H_* M$, then M can be written as a union of a filtered system of countable (but not necessarily finitely generated!) submonoids $M_i \subset M$ such that $\pi_0 M_i$ is again in the center of $H_* M_i$. Therefore, it is enough to prove 3.1 for the case where M itself is countable.

For a vertex m of M, write $\rho_m : M \to M$ for the map given by right-multiplication by m. Since $\pi_0 M$ is countable, we can pick a vertex from each component of M and arrange these vertices in a sequence $(m_i : i \in M)$ such that each element in the sequence occurs infinitely often. Now consider the homotopy colimit $\overline{M}$ of the sequence

$$M \xrightarrow{\rho_{m_1}} M \xrightarrow{\rho_{m_2}} M \longrightarrow \dots .$$

M acts on itself from the left and this action is compatible with the ρ_{m_i}'s, so M acts on $\overline{M}$. The category of elements $\overline{M}_M$ of this action is the colimit of a sequence of copies of M_M; M_M has an initial object, so its nerve $N(M_M)$ is contractible, and therefore so is $N(\overline{M}_M)$. So the homotopy fibre of $|\delta^* N(\overline{M}_M)| \to |N(M)| = BM$ is ΩBM. On the other hand, $H_*\overline{M}$ is the colimit of $H_* M \to H_* M \to \dots$ where the maps are induced by the ρ_{m_i}. So $H_*(\overline{M}) = H_*(M)[\pi_0(M)^{-1}]$ by the remark preceding the proof, and therefore M acts on $\overline{M}$ by homology equivalences (at the level of homology, right-multiplication by a vertex of M coincides with left-multiplication). By 2.1, $\overline{M}$ has the same homology as the homotopy fiber, which we have just identified as being ΩBM. This proves 3.1.

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