1 Motivation: Hopf invariant one

A division algebra structure on $\mathbb{R}^n$ is a (continuous) “multiplication” map $\mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ which is

- bilinear
- has no zero divisors – for any pair of non-zero vectors $0 \neq v, w \in \mathbb{R}^n$, $\mu(v, w) \neq 0$.

Example 1.

- The real numbers $\mathbb{R}$ with ordinary multiplication.
- The plane $\mathbb{R}^2 \cong \mathbb{C} = \{a + bi | a, b \in \mathbb{R}\}$ with multiplication given by multiplication of complex numbers.
- The four dimensional space $\mathbb{R}^4$, presented as the so-called Cayley numbers (or: Quaternions) $\mathbb{R}^4 \cong \mathbb{H} = \{a + bi + cj + dk | a, b, c, d \in \mathbb{R}\}$ with multiplication analogous to the one of complex numbers, governed by the relations $i^2 = j^2 = k^2 = -1$, $ij = k$, $ji = -k$.
- The eight dimensional space $\mathbb{R}^8$ can be given the structure of a division algebra, presented as the so-called Octonions $\mathbb{O} = \{a + bi + cj + dk + el + fm + gn + ho | a, b, c, d, e, f, g, h \in \mathbb{R}\}$ with $i^2 = j^2 = k^2 = l^2 = m^2 = n^2 = -1$, $o^2 = 1$.

Remark 1.1. Note that we did not require our multiplication $\mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$ to be associative. The Quaternions in fact associative division algebra, but the Octonions are not.

Question 1.2. Are there more? in which dimensions can we multiply vectors?.

Theorem 1.3. (Adams, Atiyah) The space $\mathbb{R}^n$ admits a structure of a division algebra, iff $n = 1, 2, 4, 8$.

Adams’ proof was the first one. It consisted of 80 pages, accessible only for a handful of experts. Using topological K-theory, Atiyah gave a very short and elegant proof for Adams theorem. To demonstrate it, he wrote it on a postcard and mailed it to a colleague!
In this course we will study define and study topological $K$-theory. We will first develop the tools of topological $K$-theory and once these will be sufficiently developed, we’ll see Atiyah’s proof, among other interesting applications.

2 Fiber bundles

We restrict our attention to compactly generated topological spaces. The main feature to have in mind is the exponential law: $\text{map}(X, \text{map}(Y, Z)) \simeq \text{map}(X \times Y, Z)$.

Let $B$ be a connected space.

Definition 2.1. A map $p : E \to B$ is called a fiber bundle with fiber $F$ if

- it is surjective.
- for every $b \in B$, there exists an open neighbourhood $U_b$ and an isomorphism of spaces, called a trivialization $\Psi_{U_b} : p^{-1}(U_b) \to U_b \times F$, compatible with the map $p$ in that the triangle

$$\begin{align*}
p^{-1}(U_b) & \xrightarrow{\sim} U_b \times F \\
& \downarrow \Psi_{U_b} \\
U_b & \to \cdot
\end{align*}$$

Remark 2.2. Thus, for any $b \in B$, $\Psi_{p^{-1}(b)} : p^{-1}(b) \to \{b\} \times F$.

Remark 2.3. You saw in the previous course that any fiber bundle is a Serre fibration.

Example 2. 1. The projection map $B \times F \to B$. This is called the trivial bundle.

2. Let $S^1 \subset \mathbb{C}$ be the unit circle. The map $p_n : S^1 \to S^1$ given by $z \mapsto z^n$ is a fiber bundle with the fiber over $1 \in S^1$ given by the set of $n$-th roots of unity.

3. The map $\exp : \mathbb{R} \to S^1$ given by $\exp(t) = e^{2\pi it} \in S^1$ is a fiber bundle with fiber $\mathbb{Z}$.

4. Recall that the $n$-dimensional real projective space is defined by $\mathbb{R}P^n = S^n/\sim$ where $x \sim x \in S^n \subset \mathbb{R}^{n+1}$. Then, the quotient map $S^n \to \mathbb{R}P^n$ is a fiber bundle with fiber $\{1, -1\}$.

5. Let $S^{2n+1} \subset \mathbb{C}^{n+1}$ and let $\mathbb{C}P^n = S^{2n+1}/\sim$ where $x \sim ux$ for any $u \in S^1$. Then the quotient map $S^{2n+1} \to \mathbb{C}P^n$ is a fiber bundle with fiber $S^1$.

6. Consider the Moebius band $M = [0, 1] \times [0, 1]/\sim$ where $(t, 0) \sim (1 - t, 1)$ and consider the “center circle” $C = (1/2, s) \subset M$. The projection map $M \to C$ given by $(t, s) \mapsto (1/2, s)$ is a fiber bundle with fiber $[0, 1]$.
Definition 2.4. Let $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ be fiber bundles. A map of fiber bundles is a commutative square

\[
\begin{array}{ccc}
E_1 & \xrightarrow{\varphi} & E_2 \\
p_1 & \downarrow & \downarrow p_2 \\
B_1 & \xrightarrow{\varphi} & B_2
\end{array}
\]

Note: such a map induces, for each $b \in B$, a map between the fibers

\[(E_1)_b \to (E_2)_{\varphi(b)}.
\]

We have thus defined the category of fiber bundles.

Observe 2.5. A map $p : E \to B$ is a covering space iff it’s a fiber bundle with discrete fiber.

3 Vector bundles

Let $k$ be either of the (topological) fields $\mathbb{R}$ or $\mathbb{C}$. We will restrict attention to finite dim’l vector spaces over $k$. Note that such a vector space $V$ is always assumed to be a topological vector space, in the sense that addition of vectors and multiplication by a scalar define continuous maps $V \times V \to V$ and $k \times V \to V$.

Definition 3.1. Let $V$ be an $n$-dim’l vector space over $k$, and let $B$ be a connected space. An $n$-dim’l vector bundle with fiber $V$ is a fiber bundle $p : E \to B$ with the structure of a vector space on each fiber $p^{-1}(b) = E_b$ such that, for each $b \in B$, the maps $\Psi_{U_b} : p^{-1}(U_b) \to U_b \times V$ restrict to $k$-linear maps (hence isomorphisms)

\[\Psi_{p^{-1}(b)} : p^{-1}(b) \xrightarrow{\cong} \{b\} \times V\]

on each fiber.

A map of vector bundles is a map of fiber bundles which is $k$-linear on each fiber. The category of vector bundles is denoted $\text{VB}$ and that of vector bundles over a fixed space $B$ is denoted $\text{VB}/B$.

Remark 3.2. We assume throughout that our base space $B$ is connected. If $B = \coprod B_\alpha$ is a disjoint union of path components, then a vector bundle $E$ over $B$ is by definition a collection of vector bundles $E_\alpha$ over each $B_\alpha$ and the rank of each $E_\alpha$ may be different. We will assume all our base spaces are connected in order to simplify the discussion. All the arguments could be extended to the case of non-connected base in a straightforward way.

Example 3.
Given an $n$-dim'l $k$-vector space $V$, the projection $B \times V \to B$ is a vector bundle, called the **trivial vector bundle**.  

The **Möebius line bundle** is given as follows. Let $E = [0,1] \times \mathbb{R}/(0,t) \sim (1,-t)$. Let $C$ be the middle circle $C = \{(s,0) \in E\}$. Then the projection $E \to C$, $(s,t) \mapsto (s,0)$ is a vector bundle with fiber $\mathbb{R}$.  

Define the **canonical line bundle** over the projective space $\mathbb{R}P^n$ as follows. The space $\mathbb{R}P^n$ may be thought of as the space of lines $\ell$ through the origin in $\mathbb{R}^{n+1}$. Let $E = (\ell,v) \in \mathbb{R}P^n$, $v \in \ell$ and define $E \to \mathbb{R}P^n$ by setting $(\ell,v) \mapsto \ell$.  

**Proposition 3.3.** Let $E \to F$ be a map of vector bundles. Then $\varphi$ is an isomorphism iff $\varphi|_{p^{-1}(b)} : p^{-1}(b) \to q^{-1}(b)$ is an isomorphism for each $b \in B$.  

**Proof.** Clearly, if $\varphi$ has a (categorical) inverse $\varphi^{-1}$, it restricts to an isomorphism on each fiber. Conversely, suppose $E = B \times V$ and $F = B \times W$ are trivial vector bundles and that $\varphi : E \to F$ restricts to an isomorphism on each fiber. By the exponential law for spaces, we have homeomorphism of spaces (with respect to the compact-open topology) 

$$\text{map}_B(B \times V, B \times W) \cong \text{map}(B \times V, W) \cong \text{map}(B, \text{map}(V,W))$$  

where $\text{map}(V,W)$ is the space of $k$-linear maps with the obvious topology.  

The (vector bundle) map $\varphi : E \to F$ thus corresponds to a map $\Phi : B \to \text{Hom}(V,W)$ which is in fact a map $\Phi : B \to \text{Iso}(V,W)$ by our assumption on $\varphi$. If we denote the (continuous) inversion map by $i : \text{Iso}(V,W) \to \text{Iso}(W,V)$ then we get the composite $\Psi = i \circ \Phi : B \to \text{Iso}(W,V)$ which by [with the roles of $V$ and $W$ interchanged] corresponds to a vector bundle map $\psi : F \to E$. The map $\psi$ is clearly an inverse to $\varphi$ since it is such on each fiber.  

Thus, the statement is true locally. If now $\varphi : E \to F$ is a map of (arbitrary) vector bundles which is an isomorphism on each fiber, then $\varphi$ is one-to-one and onto, and we need to show that its set-theoretical inverse $\varphi^{-1}$ is continuous. But $\varphi^{-1}$ coincides with $\psi$ on each piece of an open cover and we have shown that $\psi$ is continuous so $\varphi^{-1}$ must be continuous.  

$\square$
4 Sections

A section of a vector bundle \( p : E \to B \) is a map \( s : B \to E \) such that \( ps = \text{id}_B \). Thus, a section is a continuous correspondence \( b \mapsto v_b \) of a vector \( v_b \in E_b \) to each point \( b \in B \). For example, we see that every vector bundle has at least one section—the zero section \( b \mapsto 0_{E_b} \).

**Proposition 4.1.** An \( n \)-dim’l vector bundle is trivial iff it admits \( n \) linearly independent sections, i.e. sections \( \{s_1, \ldots, s_n\} \) s.t. \( \{s_1(b), \ldots, s_n(b)\} \) are linearly independent for each \( b \in B \).

**Proof.** Clearly, \( B \times \mathbb{K}^n \) has such sections, and any vector bundle isomorphism takes linearly independent sections to linearly independent sections. Conversely, if \( s_1, \ldots, s_n \) are linearly independent sections of \( p : E \to B \) then the map

\[ \varphi : B \times \mathbb{K}^n \to E \]

given by \( \varphi(b, \lambda_1, \ldots, \lambda_n) = \sum \lambda_i s_i(b) \) is an isomorphism on each fiber and hence an isomorphism of vector bundles. \( \square \)

5 Pullbacks

Let \( p : E \to B \) be a vector bundle and \( B' \to B \) any map.

**Observe 5.1.** There is an induced vector bundle structure on the pullback \( p' : E' = E \times_B B' \to B' \).

6 Direct sums

Given vector bundles \( p_1 : E_1 \to B \) and \( p_2 : E_2 \to B \), their direct sum is \( E_1 \oplus E_2 = E_1 \times_B E_2 \) together with the projection map \( p : E_1 \oplus E_2 \to B \). Note that \( p^{-1}(b) = p_1^{-1}(b) \oplus p_2^{-1}(b) \) so that the name is reasonable.

**Proposition 6.1.** The projection \( E_1 \oplus E_2 \to B \) is a vector bundle.

**Proof.** Given two vector bundles \( p_1 : E_1 \to B_1 \) and \( p_2 : E_2 \to B_2 \) the product \( p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2 \) is a vector bundle, for if \( \varphi_1 : p_1^{-1}(U_{b_1}) \xrightarrow{\sim} U_{b_1} \times V \) and \( \varphi_2 : p_2^{-1}(U_{b_2}) \xrightarrow{\sim} U_{b_2} \times W \) are trivializations, then \( \varphi_1 \times \varphi_2 : p_1^{-1}(U_{b_1}) \times p_2^{-1}(U_{b_2}) \to U_{b_1} \times U_{b_2} \times V \times W \) is a trivialization for \( E_1 \times E_2 \).

In our case, \( p_1 \times p_2 : E_1 \times E_2 \to B \times B \) is a vector bundle, and its pullback along the diagonal \( \delta : B \to B \times B \) is precisely \( E_1 \oplus E_2 \) which is therefore a vector bundle itself. \( \square \)

References


Topological K-theory, Lecture 2

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Again, we assume throughout that our base space $B$ is connected.

1 Direct sums

Recall from last time:

Given vector bundles $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$, their direct sum is $E_1 \oplus E_2 = E_1 \times_B E_2$ together with the projection map $p : E_1 \oplus E_2 \to B$. Note that $p^{-1}(b) = p_1^{-1}(b) \oplus p_2^{-1}(b)$.

Proposition 1.1. The projection $E_1 \oplus E_2 \to B$ is a vector bundle.

Example 1. The canonical line bundle on $\mathbb{R}P^n$, $E \to \mathbb{R}P^n$ has an orthogonal complement given by $E^\perp = \{(\ell, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} | v \perp \ell\}$. The map $E^\perp \to \mathbb{R}P^n$, $(\ell, v) \mapsto \ell$ is an $n$-dimensional vector bundle, whose fiber over $\ell$ is $\ell^\perp$.

Observe 1.2. We have an isomorphism of vector bundles $E \oplus E^\perp \cong \mathbb{R}P^n \times \mathbb{R}^{n+1}$ given by $(\ell, v, w) \mapsto (\ell, v + w)$. When $n = 1$, $E \to \mathbb{R}P^1 = S^1$ is the Mobius line bundle which we have shown to be non-trivial. Since in this case $E \cong E^\perp$, we see that the (direct) sum of two non-trivial bundles may be trivial. We will explore this algebraic structure more thoroughly later in the course.

2 Operations on vector bundles

Let $\text{Vect}_k$ be the category of finite dimensional vector spaces over $k (= \mathbb{R}, \mathbb{C})$. This category is enriched over topological spaces in that for every $V, W \in \text{Vect}_k$, the set of linear maps $\text{Hom}(V, W)$ admits a topology for which the composition rule is continuous.

Definition 2.1. An endofunctor $T : \text{Vect}_k \to \text{Vect}_k$ is called topological if for every $V, W \in \text{Vect}_k$, the map $T : \text{Hom}(V, W) \to \text{Hom}(TV, TW)$ is continuous.

Our goal is now to show that such a topological functor $T$ induces an endofunctor of vector bundles, obtained by applying $T$ “fiberwise”.

If $p : E \to B$ is a vector bundle, we define the set $TE$ to be the union $\bigcup_{b \in B} T(E_b)$ and if $\varphi : E \to F$ is a map of vector bundles we define the function
vector bundles. For example, we have an isomorphism $E$ constructed:

**Definition 3.1.** Let $p : B \times V \to \text{Hom}(V, W)$ and we obtain a map $T \Phi : B \to \text{Hom}(TV, TW)$ which then corresponds back to $T(\varphi) : TE \to TF$. Thus, $T(\varphi)$ is continuous because $T \Phi$ is so. Note that, if $\varphi$ is an isomorphism, then so is $T \varphi$ since in that case $T(\varphi_b)$ is an isomorphism for each $b \in B$.

Suppose $E$ is trivial but has no preferred product structure. Choose an isomorphism $\alpha : E \to B \times V$ and topologize $TE$ by requiring $T(\alpha) : TE \to B \times TV$ to be a homeomorphism (there is only one possible topology for $TE$ that make it so). If $\beta : E \to B \times V$ is any other isomorphism, then for $\varphi = \beta \alpha^{-1}$ we see that $T(\alpha)$ and $T(\beta)$ induce the same topology on $E$ since $T(\beta) = T(\varphi)T(\alpha)$ is a homeomorphism as a composition of such. We thus see that the topology on $TE$ does not depend on the choice of $\alpha$.

Furthermore, it is clear that if $\varphi : E \to F$ is a map of trivial bundles, then $T(\varphi)$ is a map of vector bundles, and that if $B \subseteq B$, $T(E)_{|B'} \cong T(E_{|B'})$ $[T(E)_{|B'} \cong B' \times TV]$.

Suppose $p : E \to B$ is arbitrary. Then if $U \subseteq B$ is such that $E_{|U}$ is trivial, we topologize $T(E_{|U})$ as above. We then topologize $TE$ by declaring a set $V \subseteq T(E)$ to be open iff $V \cap T(E_{|U})$ is open for every $U$ for which $E_{|U}$ is trivial over. As we saw last time, continuity is a local property so that for a map of arbitrary vector bundles $\varphi : E \to F$, $T(\varphi) : TE \to TF$ is continuous. If $B' \subset B$ then again $T(E_{|B'}) \cong T(E)_{|B'}$ so that the two possible topologies agree.

Let us give a few examples of the operations on vector bundles we have constructed:

i. $E \otimes F$.

ii. $\text{Hom}(E, F)$.

iii. $E^*$ — the dual bundle.

The identities these operations satisfy in vector spaces continue to hold for vector bundles. For example, we have an isomorphism $E \otimes (F^* \oplus F'') \cong (E \otimes F') \oplus (E \otimes F'')$.

### 3 Transition functions

It is common to view a vector bundle is family of vector spaces, one for every point in the base, which are glued together. We now make this precise.

**Definition 3.1.** Let $p : E \to B$ be a $k$-vector bundle with trivializations

$$E_{|U_\alpha} \xrightarrow{\varphi_\alpha} U_\alpha \times V \xrightarrow{\varphi_\beta} U_\beta \times V$$

$$B$$
that restrict to vector space isomorphisms \( \varphi_{\alpha}|_{E_b} : E_b \simarrow \{b\} \times V \). The \textbf{transition functions} are defined to be the maps

\[
g_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow \text{GL}(V) := \text{Iso}(V)
\]
given by \( g_{\beta\alpha}(b) = \varphi_{\beta}|_{E_b} (\varphi_{\alpha}|_{E_b})^{-1} \). Note that \( \text{Iso}(V) \) is a topological space since \( V \) is a topological vector space.

\textit{Observe 3.2.} The transition functions of a vector bundle satisfy the \textbf{cocycle condition}: On triple intersections \( U_\alpha \cap U_\beta \cap U_\gamma \), \( g_{\gamma\beta} g_{\beta\alpha} = g_{\gamma\alpha} \). This can be seen by the following diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\varphi^{-1}} & E_b \\
\downarrow & & \downarrow \\
\varphi & \xrightarrow{\phi} & V \\
\downarrow & & \downarrow \\
\varphi^{-1} & \xrightarrow{\phi} & E_b \\
\end{array}
\]

The previous observation admits a converse in the form of

\textbf{Proposition 3.3.} Let \( \{U_\alpha\} \) be an open cover of \( B \) and suppose we are given maps

\[
g_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow \text{GL}(V) := \text{Iso}(V)
\]
satisfying the cocycle condition. Then there is a vector bundle \( p : E \longrightarrow B \) with fiber \( V \) whose transition functions are \( g_{\beta\alpha} \).

\textit{Proof.} Define \( E := \bigsqcup_{\alpha} U_\alpha \times V / \sim \) where for every \( b \in U_\alpha \cap U_\beta \), \( (b, v) \sim (b, w) \) iff \( w = g_{\beta\alpha}(b)(v) \). The cocycle condition implies that \( g_{\alpha\beta} = g_{\beta\alpha}^{-1} \). Thus, if \( (b, v) \) is equivalent to \( (b, w) \), \( v = g_{\alpha\beta}(b)(w) \) so that \( \sim \) is symmetric. Transitivity follows in a similar way and thus \( \sim \) is an equivalence relation. Define \( p : E \longrightarrow B \) by \( p[b, v] = b \). Then the map \( U_\alpha \times V \longrightarrow \bigsqcup_{\alpha} U_\alpha \times V \longrightarrow E \) admits a factorization

\[
\begin{array}{ccc}
U_\alpha \times V & \longrightarrow & E \\
\downarrow & & \downarrow \\
E|_{U_\alpha} & \longrightarrow & E
\end{array}
\]
in which the left map is a homeomorphism. We see that \( p : E \longrightarrow B \) is a vector bundle with transition functions \( g_{\beta\alpha} \).

\section{Paracompact spaces}

Our goal for the remains of this talk and the next one is to establish a classification of vector bundles. We will need to make a mild assumption on the base space \( B \) and we review it now. The proofs of the following point-set topology assertions will be omitted. They can be found in Hatcher’s book “vector bundles and K-theory”.

\textbf{Definition 4.1.} A Hausdorff space \( X \) is \textbf{paracompact} if every open cover \( \{U_\alpha\}_{\alpha \in I} \) admits a \textbf{partition of unity} with respect to it (or: subordinated to it), i.e., there are maps \( \{h_\alpha : X \longrightarrow [0,1]\}_{\alpha \in I} \) satisfying:

\[
\begin{array}{c}
E|_{U_\alpha} \\
\end{array}
\]
Definition 4.2. An open cover \( \{U_\alpha\} \) of \( X \) is \textbf{locally finite} if for every \( x \in X \) there is an open neighbourhood \( V_x \) such that \( V_x \cap U_\alpha = \emptyset \) for almost all \( \alpha \).

There is another equivalent definition of paracompact spaces as follows

Theorem 4.3. A space \( X \) is \textbf{paracompact} iff it is Hausdorff and every open cover has a locally finite open refinement.

Finally, we need a technical

Lemma 4.4. Let \( X \) be a paracompact space. If \( \{U_\alpha\} \) is an open cover, there is a \textbf{countable} open cover \( \{V_\beta\} \) such that each \( V_\beta \) is a disjoint union of opens, each contained in some \( U_\alpha \).

5 Classification of vector bundles

Recall that we have defined (in the exercise) the Grassmanian \( G_n = G_n(k^\infty) \) to be the space of all \( n \)-dimensional subvector spaces of \( k^\infty \). We also defined \( E_n = E_n(k^\infty) = \{(V,v) \in G_n \times k^\infty| v \in V\} \) and showed that the projection map \( \gamma_n: G_n \to E_n \) given by \( (V,v) \mapsto V \) defines an \( n \)-dimensional vector bundle.

The following proposition asserts that every \( n \)-dimensional vector bundle can be obtained as a pullback along \( \gamma_n \).

Proposition 5.1. Let \( p: E \to B \) be a rank \( n \) vector bundle over \( k \) with \( B \) paracompact. Then there exists a map \( f: B \to G_n(k^\infty) \) and an isomorphism of vector bundles over \( B \), \( E \cong f^*E_n \).

Proof. We can assume that \( p: E \to B \) has trivializations \( \varphi_\alpha : p^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times V \) with \( \{U_\alpha\}_{\alpha \in I} \) locally finite and countable. Let \( \{h_\alpha: X \to [0,1]\} \) be a partition of unity wrt \( \{U_\alpha\} \) and define \( g_\alpha: E \to V \) by \( g_\alpha|_{p^{-1}(U_\alpha)} = (h_\alpha p) \circ (\pi_2 \varphi_\alpha) \) (where \( \pi_2 : U_\alpha \times V \to V \) is the projection map) and \( g_\alpha = 0 \) else. Note that \( g_\alpha \) is continuous since \( h_\alpha|_{0}^{-1}(0,1] \subseteq U_\alpha \). Choose an isomorphism \( \Sigma_\alpha V \cong k^\infty \) (\( I \) is countable) and define \( g = \Sigma_\alpha g_\alpha : E \to \Sigma_\alpha V \cong k^\infty \). Then \( g \) is well-defined since \( \{U_\alpha\} \) is locally finite. We now claim that \( g \) maps each \( E_b \) isomorphically onto \( V \). This is so since if \( h_\alpha(b) \neq 0 \) then for any \( e \in E_b \), \( g(e) = \Sigma_\alpha g_\alpha(e) = (\Sigma_\alpha h_\alpha(b)) \cdot (\pi_2 \varphi_\alpha(e)) = \pi_2(\varphi_\alpha(e)) \in V \). Define \( f: B \to G_n(k^\infty) \) via \( f(b) = g(E_b) \).
We consider the pullback

\[
\begin{array}{ccc}
  f^*(E_n(k^\infty)) & \longrightarrow & E_n(k^\infty) \\
  \downarrow & & \downarrow \\
  B & \longrightarrow & G_n(k^\infty).
\end{array}
\]

Then \( f^* E_n(k^\infty) \) consists of triples \((b, V, v)\) such that \( g \) maps \( E_b \) isomorphically onto \( V \subseteq k^\infty \). Thus, the map \( E \rightarrow f^*(E_n(k^\infty)) \) given by the isomorphism \( g : E_b \rightarrow V \) on every fiber \( E_b \) is an isomorphism of vector bundles. \qed
Recall that we assume throughout our base space $B$ is connected.

## 1 Classification of vector bundles – continued

Last time we showed that any $n$-dimensional vector bundle $p : E \rightarrow B$ is isomorphic to a vector bundle obtained via pulling back the bundle $\gamma_n : E_n \rightarrow G_n$ as depicted below

$$
\begin{array}{cccc}
E & \xrightarrow{z} & f^*(E_n(k^\infty)) & \xrightarrow{\tilde{f}} & E_n(k^\infty)k \xrightarrow{\pi} k^\infty \\
\downarrow_p & & \downarrow & & \downarrow_{\gamma_n} \\
B & \xrightarrow{f} & G_n(k^\infty).
\end{array}
$$

Furthermore, in such a setting, one can see that the map $\pi\tilde{f}_\alpha$ is a linear injection on each fiber.

The following Theorem will be a key in the second part of the classification:

**Theorem 1.1.** Let $B$ be paracompact and let $p : E \rightarrow B \times I$ be a vector bundle. Then $E|_{X \times \{0\}} \cong E|_{X \times \{1\}}$.

We first prove a couple of lemmas

**Lemma 1.2.** Let $B$ be paracompact. A vector bundle $p : E \rightarrow B \times I$ whose restrictions over $B \times [0,t]$ and over $B \times [t,1]$ are trivial is trivial as well.

**Proof.** Let $h_0 : E_0 := E|_{B \times [0,t]} \xrightarrow{z} B \times [0,t] \times V$, and $h_1 : E_1 := E|_{B \times [t,1]} \xrightarrow{z} B \times [t,1] \times V$ be isomorphisms to trivial bundles. The maps $h_0, h_1$ may not agree on $E|_{B \times \{t\}}$ so we cannot yet glue them. Define an isomorphism $h_{01} : B \times [t,1] \times V \rightarrow B \times [t,1] \times V$ by duplicating the map $h_0h_1^{-1} : B \times \{t\} \times V \rightarrow B \times \{t\} \times V$ on each slice $B \times \{s\} \times V$ for $t \leq s \leq 1$ and set $\overline{h_1} := h_{01}h_1$. Then $\overline{h_1}$ is an isomorphism of bundles and agrees with $h_0$ on $E|_{B \times \{t\}}$, we can now glue together $h_0$ and $h_1$ to get the desired. $\square$

**Lemma 1.3.** For every vector bundle $p : E \rightarrow B \times I$ there is an open cover $\{U_\alpha\}_\alpha$ such that each restriction $E|_{U_\alpha \times I} \rightarrow U_\alpha \times I$ is trivial.
Proof. For each $b \in B$, take open neighbourhoods $U_b$ with $0 = t_0 < t_1 < \ldots < t_k = 1$ such that $E|_{U_b \times [t_{i-1}, t_i]} \rightarrow U_b \times [t_{i-1}, t_i]$ is trivial. This is possible because for each $(b, t)$ we can find an open neighbourhood of the form $U_b \times J_i$ (where $J_i$ is an open interval) over which $E$ is trivial; if we then fix $b$ then the collection $\{J_i\}$ covers $I$ and we can take a finite subcover $J_1, \ldots, J_{k+1}$ and choose $t_i \in J_i \cap J_{i+1}$, this way $E$ remains trivial over $U_b \times [t_{i-1}, t_i]$. Now, by Lemma 1.2 $E$ is trivial over $U_b \times I$. 

Proof of Theorem 1.1. By Lemma 1.3, take an open cover $\{U_a\}_a$ of $B$ such that $E|_{U_a \times I}$ is trivial. Assume first that $B$ is compact. Then we can take a cover of the form $\{U_i\}_{i=1}^n$. Take a partition of unity $\{h_i : B \rightarrow I\}_{i=1}^n$ subordinated to $\{U_i\}$. For $i \geq 0$, set $g_i = h_1 + \ldots + h_i \ (g_0 = 0, \ g_n = 1)$, let $B_i = Gr(g_i) \subseteq B \times I$ be the graph of $g_i$ and let $p_i : E_i \rightarrow B_i$ be the restriction of $E$ to $B_i$. The map $B_i \rightarrow B_{i-1}$ given by $(b, g_i(b)) \mapsto (b, g_{i-1}(b))$ is a homeomorphism, and since $E|_{U_i \times I}$ is trivial, the dotted isomorphism in the diagram below exists:

\[ E|_{B_i \cap (U_i \times I)} \sim \cdots \sim E|_{B_{i-1} \cap (U_i \times I)} \]

\[ B_i \cap (U_i \times I) \stackrel{z}{\longrightarrow} B_{i-1} \cap (U_i \times I) \]

(specifically: a restriction of a trivial bundle is trivial, and trivial bundles over homeomorphic bases are isomorphic.) Since outside $U_i$, $h_i = 0$, $E|_{B_i \cap U_i^c}$, we obtain an isomorphism of vector bundles (over different bases) $f_i : E|_{B_i} \sim E|_{B_{i-1}}$. The composition $f = f_1 \circ \ldots \circ f_n$ is then an isomorphism from $E|_{B_n} = E|_{B \times \{1\}}$ to $E|_{B_0} = E|_{B \times \{0\}}$.

Assume now $B$ is paracompact. Take a countable cover $\{V_i\}_i$ such that each $V_i$ is a disjoint union of opens, each contained in some $U_a$. This means that $E$ is trivial over each $V_i \times I$. Let $\{h_i : B \rightarrow I\}$ be a partition of unity subordinated to $\{V_i\}_i$ and set as before $g_i := h_1 + \ldots + h_i$ and $p_i : E_i \rightarrow B_i := Gr(g_i)$ the restriction. As before, we obtain isomorphisms $f_i : E_i \sim E_{i+1}$. The infinite composition $f = f_1 f_2 \ldots$ is well-defined since for every point, almost all $f_i$’s are the identity. As before, $f$ is an isomorphism from $E|_{B \times \{1\}}$ to $E|_{B \times \{0\}}$. 

In other words, Theorem 1.1 tells us that homotopic maps induce isomorphic pullback bundles. Let $VBun_k^n(B)$ be the set of isomorphism classes of rank $n$ $k$-vector bundles over $B$.

Corollary 1.4. A homotopy equivalence of paracompact spaces $f : A \rightarrow B$ induces a bijection $f^* : VBun_k^n(B) \sim VBun_k^n(A)$. In particular, any vector bundle over a contractible paracompact space is trivial.

Proof. If $g$ is a homotopy inverse of $f$, then $f^* g^* = id^* = id$ and $g^* f^* = id^* = id$. 

□
We are ready to state and prove the classification theorem.

**Theorem 1.5.** Let $B$ be paracompact. Then pullback along $\gamma_n : E_n(\mathbf{k}^\infty) \to G_n(\mathbf{k}^\infty)$ induces a bijection

\[
[B, G_n(\mathbf{k}^\infty)] \xrightarrow{\sim} \text{VBun}_n^B(B)
\]

\[
[f] \mapsto f^* E_n
\]

**Proof.** The map is well-defined since two homotopic maps give two isomorphic pullback vector bundles. Proposition 5.1 of Lecture 2 gives surjectivity of $\Box$ so we are left with injectivity.

Assume we have two maps $f_0, f_1 : B \to G_n(\mathbf{k}^\infty)$ which induce isomorphic bundles upon pullback. It would be convenient to assume we are given a vector bundle $p : E \to B$ and a couple of isomorphisms of bundles $i_0 : E \xrightarrow{\sim} f_0^* E_n$ and $i_1 : E \xrightarrow{\sim} f_1^* E_n$ so that we have the following commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
E & \xrightarrow{i_0} & f_0^* E_n \\
\downarrow p & & \downarrow f_0 \\
B & \xrightarrow{f_0} & G_n(\mathbf{k}^\infty)
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
E & \xrightarrow{i_1} & f_1^* E_n \\
\downarrow p & & \downarrow f_1 \\
B & \xrightarrow{f_1} & G_n(\mathbf{k}^\infty)
\end{array}
\end{array}
\]

in which the maps $f_0$ and $f_1$ are obtained as pullbacks of $f_0$ and $f_1$ respectively.

Define $g_0 := \pi f_0 i_0$ and $g_1 := \pi f_1 i_1$. Then $g_0, g_1 : E \to \mathbf{k}^\infty$ are linear injections on each fiber and satisfy $f_0(b) = g_0(E_b)$ and $f_1(b) = g_1(E_b)$. It will thus be enough to find a homotopy $\{g_t\}$ from $g_0$ to $g_1$ in which all maps $g_t$ are linear injections on each fiber since we could then define $f_t(b) = g_t(E_b) \in G_n(\mathbf{k}^\infty)$ to obtain a homotopy from $f_0$ to $f_1$.

Composing $g_0$ with the maps $L_t : \mathbf{k}^\infty \to \mathbf{k}^\infty$ given by

\[
(v_1, v_2, ...) \mapsto (1 - t)(v_1, v_2, ...) + t(v_1, 0, v_2, 0, ...)
\]

gives a homotopy from $g_0$ to a map $\overline{g_0}$ through maps which are linear injections on each fiber. The image of $\overline{g_0}$ lies in the subspace of $\mathbf{k}^\infty$ consisting of vectors with non-zero components only in the odd coordinates. Similarly, we can replace $g_1$ by a map $\overline{g_1} : E \to \mathbf{k}^\infty$ whose image lies in the subspace of $\mathbf{k}^\infty$ consisting of vectors with non-zero components only in the even coordinates. Clearly, it is enough to construct a homotopy $\{\overline{g}_t\}$ from $\overline{g_0}$ to $\overline{g_1}$ through maps which are linear injections on each fiber. But this is easy now: we set $\overline{g}_t := (1 - t)\overline{g_0} + t\overline{g_1}$ and finish the argument.

\[\square\]
Theorem 1.5 justifies the following terminology.

**Definition 1.6.** The bundle $\gamma_n : E_n(k^\infty) \to G_n(k^\infty)$ is called the **universal rank-$n$ vector bundle**.

## 2 Applications of the classification theorem

Let us see how the classification theorem can be used.

**Example 1.** The bundle $\gamma_n : E_n(k^\infty) \to G_n(k^\infty)$ admits an inner product, induced from an inner product on $k^\infty$. Since every rank-$n$ vector bundle is obtained as a pullback along $\gamma_n$, we deduce that any vector bundle admits an inner product – that obtained by pulling back the one on $\gamma_n$. This is a shortened proof to what you already showed in the exercise.

**Example 2.** Let us compute the Picard group of complex projective spaces. By the classification theorem we have

$$\text{VBun}_C^1(CP^n) \cong [CP^n, G_1(C^\infty)] = [CP^n, CP^\infty].$$

You have shown in the exercise that $V_1(C^\infty) \to G_1(C^\infty)$ is a fiber bundle with fiber $\text{GL}_1(C)$. By a theorem you proved in Algebraic topology I, any fiber bundle is a Serre fibration, so that we have a fibration sequence

$$\text{GL}_1(C) \to V_1(C^\infty) \to G_1(C^\infty).$$

You have seen in the exercise class that the spaces $V_n(k^\infty)$ are contractible and it is easy to see that $\text{GL}_1(C) \cong S^1$ (this is so since $\text{GL}_1(C) = C \setminus \{0\}$). Thus, the long exact sequence for the fibration sequence\(^3\) implies that $G_1(C^\infty) = CP^\infty$ is a $K(\mathbb{Z}, 2)$. Now, an application of Brown’s representability theorem (which you proved in Algebraic Topology I) implies that $[CP^n, CP^\infty] = [CP^n, K(\mathbb{Z}, 2)] \cong H^2(CP^n; \mathbb{Z})$ – i.e. the second cohomology group of $CP^n$. Using cellular cohomology (this is an elementary way of calculation, given in any first course in cohomology) we deduce from the cell structure of $CP^n$ (one cell in each even dimension and no others) that $\text{Pic}(CP^n) = \text{VBun}_C^1(CP^n) \cong H^2(CP^n; \mathbb{Z}) \cong \mathbb{Z}$.

In fact, it follows from what you showed in the exercise that group structure is given by the tensor product. Thus, there is a line bundle $\zeta$ on $CP^n$ (corresponding to $1 \in \mathbb{Z}$) such that $\zeta \otimes \ldots \otimes \zeta$ $(n$-times) correspond to $n \in \mathbb{Z}$ – this is the canonical line bundle introduced in Lecture 1!
1 Applications of the classification theorem – continued

Let us see how the classification theorem can further be used.

Example 1. The bundle $\gamma_n : E_n(k^\infty) \rightarrow G_n(k^\infty)$ admits an inner product, induced from an inner product on $k^\infty$. Since every rank-$n$ vector bundle is obtained as a pullback along $\gamma_n$, we deduce that any vector bundle admits an inner product – that obtained by pulling back the one on $\gamma_n$. This is a shortened proof to what you already showed in the exercise.

Example 2. Let us compute the Picard group of real projective spaces. By the classification theorem we have $\text{VBun}^1_{\mathbb{R}}(\mathbb{R}P^n) \cong [\mathbb{R}P^n, G_1(\mathbb{R}^\infty)] = [\mathbb{R}P^n, \mathbb{R}P^\infty]$. You have shown in the exercise that $\text{VBun}^1_{\mathbb{R}}(\mathbb{R}^\infty) \rightarrow G_1(\mathbb{R}^\infty)$ is a fiber bundle with fiber $\text{GL}_1(\mathbb{R})$. By a theorem you proved in Algebraic topology I, any fiber bundle is a Serre fibration, so that we have a fibration sequence

$$\text{GL}_1(\mathbb{R}) \rightarrow V_1(\mathbb{R}^\infty) \rightarrow G_1(\mathbb{R}^\infty).$$

You have seen in the exercise class that the spaces $V_n(k^\infty)$ are contractible and it is easy to see that $\text{GL}_1(\mathbb{R}) \cong \mathbb{Z}/2$. Thus, the long exact sequence for the fibration sequence implies that $G_1(\mathbb{R}^\infty) = \mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2, 1)$. Now, an application of Brown’s representability theorem (which you proved in Algebraic Topology I) implies that $[\mathbb{R}P^n, \mathbb{R}P^\infty] = [\mathbb{R}P^n, K(\mathbb{Z}/2, 1)] \cong H^1(\mathbb{R}P^n; \mathbb{Z}/2)$ – i.e. the first cohomology group of $\mathbb{R}P^n$ with coefficients in $\mathbb{Z}/2$. Using cellular cohomology (this is an elementary way of calculation, given in any first course in cohomology) we deduce that $\text{Pic}(\mathbb{R}P^n) = \text{VBun}^1_{\mathbb{R}}(\mathbb{R}P^n) \cong H^1(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$. In fact, it follows from what you showed in the exercise that group structure is given by the tensor product. Thus, there is a line bundle $\zeta$ on $\mathbb{C}P^n$ (corresponding to $1 \in \mathbb{Z}$) such that $\zeta \otimes \ldots \otimes \zeta$ (n-times) correspond to $n \in \mathbb{Z}$ – this is the canonical line bundle introduced in Lecture 1!

Observe 1.1. Let $B$ be a paracompact space. Then any $n$-dimensional bundle can be embedded in a trivial infinite bundle.
Proof. Write

\[
\begin{array}{cccc}
  E & \xrightarrow{z} & f^*(E_n(k^\infty)) & \xrightarrow{\tilde{f}} \ xrightarrow{f} \ xrightarrow{\gamma_n} G_n(k^\infty) \\
  & \downarrow p & & \downarrow f \hspace{1cm} G_n(k^\infty).
\end{array}
\]

We saw last time that \(\tilde{f}\) is a linear injection on fibers and to the composite \(E \longrightarrow G_n(k^\infty) \times k^\infty\) has the same property.

**Corollary 1.2.** If \(B\) is compact Hausdorff, any \(n\)-dimensional vector bundle can be embedded in a trivial (finite dimensional) bundle.

Proof. For \(d > n\),

\[
G_n(k^d) \subseteq G_n(k^{d+1}) \subseteq \cdots \subseteq \bigcup_{d > n} G_n(k^d) = G_n(k^\infty).
\]

Since \(B\) is compact, the classifying map \(B \longrightarrow G_n(k^\infty)\) factors as \(B \xrightarrow{f'} G_n(k^d) \xrightarrow{i} G_n(k^\infty)\) and by the pasting lemma for pullbacks we get that \(f^*E_n(k^\infty) \cong f'^*E_n(k^d)\). We thus get

\[
\begin{array}{cccc}
  E & \xrightarrow{z} & f'^*(E_n(k^d)) & \xrightarrow{\tilde{f}} \ xrightarrow{f'} \ xrightarrow{\gamma_n} G_n(k^d) \\
  & \downarrow p & & \downarrow f' \hspace{1cm} G_n(k^d).
\end{array}
\]

i.e. an embedding of \(E\) in a trivial bundle.

\[
\square
\]

## 2 K-theory

Corollary 1.2 is going to be crucial for us. We thus assume throughout that all our spaces are compact Hausdorff. This includes for example all finite CW-complexes.

Let \(B\) be a connected space. Denote by \(\text{VBun}^n_k(B)\) the set of isomorphism classes of \(n\)-dimensional vector bundles over \(B\). Set \(\text{VBun}^n_k(B) = \bigoplus_{n \geq 0} \text{VBun}^n_k(B)\) where by convention \(\text{VBun}^0_k(B) = \ast\). The direct sum of vector bundles induces an abelian monoid structure on \(\text{VBun}^n_k(B)\). We can further extend this by setting, for a non-connected space \(B \sqcup_{\alpha} B_{\alpha}\) (a disjoint union of path components), \(\text{VBun}(B) = \prod_{\alpha} \text{VBun}^*(B_{\alpha})\) with the ordinary abelian monoid structure.
2.1 Group completion

We would like to turn the abelian monoid $\text{VBun}_k^*(B)$ (or $\text{VBun}(B)$) into an
abelian group so that we could apply group theoretic methods in calculations.
Of course, we need some canonical way to do so, and we can obtain such by
requiring a universal property. The following is a purely algebraic method.

**Definition 2.1.** Let $A$ be an abelian monoid. A *group completion* of $A$ is an
abelian group $K(A)$ together with a map of (abelian) monoids $\alpha = \alpha_A : A \rightarrow K(A)$ such that for any abelian group $A'$ and any map of abelian monoids
$\rho : A \rightarrow A'$, there exists a unique map of abelian groups $\overline{\rho} : K(A) \rightarrow A'$
rendering the following diagram commutative:

\[
\begin{array}{ccc}
A & \rightarrow & K(A) \\
\downarrow{\rho} & & \downarrow{\overline{\rho}} \\
A' & & \\
\end{array}
\]

**Remark 2.2.** Clearly, if $K(A)$ exists, it is unique up to a unique isomorphism.

Let us construct $K(A)$ for an arbitrary $(A, \oplus)$. Let $F(A)$ be the free abelian
group generated by the underlying set of $A$ and let $E(A) \subseteq F(A)$ be the sub-
group generated by elements of the form $a + a' - a \oplus a'$ where $+ = +_{F(A)}$. The
quotient $K(A) := F(A)/E(A)$ is clearly an abelian group which together with
the obvious map $\alpha : A \rightarrow K(A)$ satisfy the universal property of $[2]$

Alternatively we can define $K(A)$ as follows. Let $\Delta : A \rightarrow A \times A$ be
the diagonal. The quotient $K(A) = A \times A/\Delta(A)$ inherits an abelian monoid
structure which has inverses since $[a, a] = 0$. We think of an element $[a, b]$ of
$K(A)$ as a formal difference $a - b$ where $[a, b] = [a', b']$ iff $a \oplus b' = a' \oplus b$. We set
$\alpha_A : A \rightarrow K(A)$ by $a \mapsto [a, 0]$. Since $K(A)$ is functorial in $A$, we get for any
map of abelian monoids $\rho : A \rightarrow B$, a commutative square

\[
\begin{array}{ccc}
A & \rightarrow & K(A) \\
\downarrow{\rho} & & \downarrow{K(\rho)} \\
B & \rightarrow & K(B) \\
\end{array}
\]

If $B$ is in fact an abelian group, $\alpha_B$ is an isomorphism so that $\overline{\rho} := \alpha_B^{-1} \circ K(\rho)$
 satisfy the universal property.

**Exercise 2.3.**

- Let $\text{AbGp}$ and $\text{AbMon}$ be the categories of abelian groups and abelian monoids
  respectively. Show that there is an adjunction

$$K : \text{AbMon} \xrightarrow{\perp} \text{AbGp} : U$$

where $U$ is the forgetful functor.
Show that if $A$ was a (commutative) semi-ring (i.e. admits a commutative ‘multiplication’ operation $\otimes$ which distributes over $\oplus$) then $K(A)$ is in fact a commutative ring.

Recall that our assumption throughout was that our base space $B$ is connected.

**Definition 2.4.** The $K$-groups of a connected space $B$ are defined to be

$$K(B) = KU(B) := K(VBun_{\mathbb{C}}(B), \oplus)$$

and

$$KO(B) = K(VBun_{\mathbb{R}}(B)).$$

where the monoid structure is taken to be direct sum of vector bundles and the ring structure is induced from tensor product of bundles.

If $B = \bigsqcup_a B_a$ a disjoint union of path components, we set

$$K(B) = K(VBun(B))$$

and similarly for $KO(B)$.

From now on, we will focus on $K(B)$ but almost everything works equally well for $KO(B)$.

Let $\mathbb{Ch}$ be the category of compact Hausdorff spaces. The assignment $B \mapsto K(B)$ defines a functor $K : \mathbb{Ch}^{op} \rightarrow AbGp$ by setting for a map $B' \rightarrow B$, $K(f) := f^* : K(B) \rightarrow K(B')$. The pasting lemma for pullbacks verifies $(f \circ f')^* = f'^* \circ f^*$.

**Observe 2.5.** Using our second construction of $K$, an element of $K(B)$ can be described as a formal difference $[E] - [F]$ of isomorphism classes of vector bundles. The expression is sometimes called a virtual vector bundle.

Let $\tau_n$ denote the trivial bundle of rank $n$.

**Observe 2.6.** If $E$ is a vector bundle over $B$, there is $n \in \mathbb{N}$ and an embedding $E \rightarrow \tau_n$ (i.e. a map of vector bundles which is linear injection on each fiber). Then we can take the orthogonal complement $E^\perp$ of $E$ with respect to $\tau_n$. This is done just as the other operations on vector bundles we talked about before – fiberwise. Strictly speaking, $(\cdot)^\perp$ is not a functor on finite dimensional vector spaces but rather a (topological) functor on finite dimensional vector spaces, embedded in some ambient vector space. The induced functor on (suitable) vector bundles is constructed in the same way as before.

It follows that for any $E$ there is an $n \in \mathbb{N}$ such that $E \oplus E^\perp \cong \tau_n$.

Suppose $[E] - [F] \in K(B)$ and let $G$ be a vector bundle such that $F \oplus G$ is trivial. Then


Thus every element in $K(B)$ is of the form $[H] - [\tau_n]$. Suppose $[E] = [F]$ in $K(B)$. Then $([E],[F]) = ([G],[G])$ for some $G$ so that $E \oplus G \cong F \oplus G$. Let $G'$ be such that $G \oplus G' \cong \tau_n$. Then $E \oplus \tau_n \cong F \oplus \tau_n$. We would like to view all trivial as one (trivial) element. We thus make the following
Definition 2.7. Two vector bundles $E$ and $F$ over $B$ are said to be stably equivalent if there are $m, n \in \mathbb{N}$ such that $E \oplus \tau_n \cong F \oplus \tau_m$.

We denote by $\simeq_S$ the equivalence relation of stably equivalent vector bundles and let $\text{SVBun}_k^*(B) := \text{VBun}_k^*(B)/\simeq_S$.

Suppose now $B$ is pointed, namely equipped with a map $* \to B$. We obtain an augmentation map $\epsilon : K(B) \to K(*) \cong \mathbb{Z}$.

Definition 2.8. The reduced $K$-theory of a pointed space $(B, *)$ is defined to be $\tilde{K}(B) = \ker(\epsilon : K(B) \to K(*) )$.

The map $\epsilon : K(B) \to \mathbb{Z}$ is given by $[E] \mapsto \dim E$. It follows that $\tilde{K}(B)$ consists of elements of the form $[E] - [F]$ with $\dim E = \dim F$.

Observe 2.9. The map $B \to *$ gives a natural splitting $K(B) \cong \tilde{K}(B) \oplus \mathbb{Z}$.

The following proposition shows that the algebraic description of Definition 2.8 is equivalent to the geometric description of Definition 2.7.

Proposition 2.10. Let $(B, *)$ be a pointed (compact) space. Then $\text{SVBun}^*(B)$ is an abelian group, and there is an isomorphism $\text{SVBun}^*(B) \cong \tilde{K}(B)$.

Proof. Clearly, $\text{SVBun}^*(B)$ is an abelian monoid under direct sum and has inverses since the isomorphism $E \oplus E^\perp \cong \tau_n$ implies $[E]^{-1} \cong [E^\perp]$.

The natural surjection $\text{VBun}^*(B) \to \text{SVBun}^*(B)$ is a map into an abelian group and the universal property of $K$ implies the existence of the dashed arrow $\rho$, which must also be a surjection:

![Diagram](https://via.placeholder.com/150)

Here, the map $K(B) \to \tilde{K}(B)$ is given by $[E] \mapsto [E] - [\tau_{\dim E}]$ (recall that elements in $\tilde{K}(B)$ are of the form $[E] - [F]$ with $\dim E = \dim F$).

Since $\rho(\tau_n) = 0$, we get a factorization of $\rho$ through the map $f : \tilde{K}(B) \to \text{SVBun}^*(B)$ given by $[E] - [F] \mapsto [E] - [F]_S$. The map $f$ is surjective since $\rho$ is.

To prove injectivity of $f$, we construct a left inverse. The map $\text{VBun}^*(B) \to K(B) \to \tilde{K}(B)$ given by $[E] \mapsto [E] - [\tau_n]$ respects $\simeq_S$ and hence induces a map $j : \text{SVBun}^*(B) \to \tilde{K}(B)$. If $[E] - [F] \in \tilde{K}(B)$ then $j(f([E] - [F])) = [E] - [\tau_n] - ([F] - [\tau_n])$ since $\dim E = \dim F$. We see that $jf = \text{id}$ so that $f$ is injective and hence an isomorphism.

3 K-theory as a generalized cohomology theory

If $B = B' \sqcup B'' \in \mathcal{CH}$ (note that it must be a finite disjoint union because of compactness), we have $\text{VBun}^*(B) = \text{VBun}^*(B') \oplus \text{VBun}^*(B'')$. Since $\oplus$ is the
coproduct in both $\text{AbMon}$ and $\text{AbGp}$ and $K$ is a left adjoint, $K(B) = K(B') \oplus K(B'')$. Let $(-)^*$ be the left adjoint to the forgetful functor $\text{CH}_* \to \text{CH}$ from pointed compact Hausdorff spaces (and pointed maps) to compact Hausdorff spaces. It is given by $B^* := B \bigsqcup \{\ast\}$. We then have $K(B^*) = \ker(e : K(B) \oplus K(\ast) \to K(\ast)) = K(B)$. For an inclusion $i : B' \to B$ in $\text{CH}$ we make a

**Definition 3.1.** The relative K-groups of a pair $B' \subseteq B \in \text{CH}$ are $K(B, B') := \tilde{K}(B/B')$ where the base-point is taken to be $B'/B'$.

We have $K(B, \varnothing) = K(B^* = K(B))$ so that our new definition specializes to the old one in the degenerate case. Our aim now is to establish an exact sequence of the form

$$K(B, B') \to K(B) \to K(B')$$

for any pair $B' \subseteq B \in \text{CH}$. In order to do this, we need to be able to construct vector bundles on $B/B'$ from vector bundles on $B$ which are trivial on $B'$.

### 3.1 Construction of bundles over quotients

We assume that $B' \subseteq B \in \text{CH}$ is a pair and denote by $q : B \to B/B'$ the quotient map. Suppose $p : E \to B$ is a vector bundle which is trivial over $B'$. Let $\alpha : E|_{B'} \to B' \times V$ be a trivialization and let $\pi : B' \times V \to V$ be the projection. Define an equivalence relation on $E|_{B'}$ by setting $e \sim e'$ iff $\pi(\alpha(e)) = \pi(\alpha(e'))$ and extend this relation by identity to $E$. Let $E/\alpha := E/\sim$ be the quotient space and set $\overline{p} : E/\alpha \to B/B'$ by $\overline{p}(\alpha) = q(\alpha)$. Note that $\overline{p}$ is well-defined since if $e \neq e'$, $e \sim e'$ only if $p(e), p(e') \in B'$. In fact, $e \sim e'$ only if they are in a different fiber which means that we collapsed all the fibers parametrized by $B'$ into a single fiber. Thus, $\overline{p} : E/\alpha \to B/B'$ has a fiber isomorphic to $V$ over every point. We would like to show that $\overline{p} : E/\alpha \to B/B'$ is in fact a fiber bundle. For that will will need a lemma which you are requested to prove in the exercise.

**Lemma 3.2.** If $E \to B$ is trivial over a closed subspace $B' \subseteq B$ then there exists an open neighbourhood $B' \subseteq U \subseteq B$ over which $E$ is still trivial.

Take such an open $B' \subseteq U$ and a trivialization $(\varphi_1, \varphi_2) : E|_U \to U \times V$. Then this induces a trivialization $\varphi : (E|_U)/\alpha \to (U/B') \times V$ given by $\varphi([e]) = (q\varphi_1(e), \varphi_2(e))$. This is a local trivialization of $E/\alpha$ around $B'/B' \in B/B'$. Around $b \in B - B'$ we have an open neighbourhood $U \subseteq B - B'$ so that we can use the same local trivializations of $E \to B$ (restricted to $U$) to get a trivialization of $E/\alpha \to B/B'$. We deduce that $E/\alpha \to B/B'$ is a vector bundle.
LECTURE 5: HIGHER K-THEORY GROUPS

In this lecture we are going to define the higher (negative) $K$-groups of a space and see a few properties of them. We have defined $K(X)$ as the group completion of the abelian monoid of isomorphism classes of vector bundles over $X$. In fact, $K(X)$ is $K^0(X)$ in an infinite sequence of abelian groups $K^n(X)$ for $n \in \mathbb{Z}$. Our aim is to show see that this sequence defines a cohomology theory in the sense of Eilenberg–Steenrod. In order to define the higher $K$-theory groups we need to introduce first some notation and topological constructions.

5.1. Notation and basic constructions

5.1.1. Let $\text{Top}$ denote the category of compact Hausdorff spaces and $\text{Top}_*$ the category of pointed compact Hausdorff spaces. By $\text{Top}_2$ we denote the category of compact pairs, that is, the objects are pairs of spaces $(X,A)$, where $X$ is compact Hausdorff and $A \subseteq X$ is closed. There are functors

$$\text{Top} \longrightarrow \text{Top}_2 \quad \text{and} \quad \text{Top}_2 \longrightarrow \text{Top}_*$$

where the basepoint in the quotient $X/A$ is $A/A$. If $A = \emptyset$, then $X/\emptyset = X$ is the space $X$ with a disjoint basepoint.

5.1.2. In what follows, we will consider complex vector bundles although most of the theory works the same in the real case. Recall that for a space $X$ in $\text{Top}$ we denote by $K(X)$ the group completion of $\text{Vect}_c(X)$. For a pointed space $X$ in $\text{Top}_*$, the reduced $K$-theory group $\widetilde{K}(X)$ is the kernel of $i^*: K(X) \rightarrow K(x_0) = \mathbb{Z}$, where $i^*$ is the map induced by the inclusion of the basepoint $i: x_0 \rightarrow X$. There is a short exact sequence

$$0 \rightarrow \ker i^* = \widetilde{K}(X) \rightarrow K(X) \xrightarrow{i^*} K(x_0) \rightarrow 0$$

which has a section $c^*$ induced by the unique map $c: X \rightarrow x_0$. So it gives a natural splitting $K(X) \cong \widetilde{K}(X) \oplus K(x_0)$. We also have that $K(X) = K(X_+)$ for every $X$ in $\text{Top}$. Hence $\widetilde{K}$ defines a contravariant functor from $\text{Top}_*$ to abelian groups.

For a compact pair $(X,A)$, we define $K(X,A) = \widetilde{K}(X/A)$. So $K(-,-)$ is a contravariant functor from $\text{Top}^2$ to abelian groups.

5.1.3. Recall that the smash product of two pointed spaces is defined as the quotient $X \wedge Y = X \times Y / X \vee Y$, where $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$ is the wedge of $X$ and $Y$, that is, the disjoint union glued by the basepoints.

We will use as a model for the $n$th sphere $S^n$ in $\text{Top}_*$ the space $I^n/\partial I^n$, where $I = [0,1]$. There is a homeomorphism $S^n \cong S^1 \wedge \cdots \wedge S^1$.

For a pointed space $X$ in $\text{Top}_*$, we define the reduced suspension $\Sigma X$ as $S^1 \wedge X$. The $n$th reduced suspension of $X$ is then $\Sigma^n X = S^n \wedge X$.
5.2. Negative $K$-groups

We can use the reduced suspension to define the negative $K$-groups for spaces, pointed spaces and pairs of spaces.

**Definition 5.2.1.** Let $n \geq 0$. For $X$ in $\text{Top}_+$, we define $\widetilde{K}^{-n}(X) = \widetilde{K}(\Sigma^n X)$. If $(X, A)$ is in $\text{Top}^2$, then we define $K^{-n}(X, A) = \widetilde{K}^{-n}(X/A) = \widetilde{K}(\Sigma^n(X/A))$. Finally, for $X$ in $\text{Top}$, we define $K^{-n}(X) = K^{-n}(X, \emptyset) = \widetilde{K}^{-n}(X+) = \widetilde{K}(\Sigma^n(X+))$.

Thus, $\widetilde{K}^{-n}(\cdot)$, $K^{-n}(\cdot, \cdot)$ and $K^{-n}(\cdot)$ are contravariant functors for every $n \geq 0$ from $\text{Top}_+$, $\text{Top}^2$ and $\text{Top}$, respectively, to abelian groups.

5.2.2. Another useful construction is the cone on a space. Given $X$ in $\text{Top}$, we define the cone on $X$ as the quotient $C_u X = X \times I / X \times \{0\}$. The cone $C_u X$ has a natural basepoint given by $X \times \{0\}$ and thus defines a functor $C: \text{Top} \rightarrow \text{Top}_+$. The quotient $C_u X / X$ is called the unreduced suspension of $X$.

If $X$ is a pointed space, then we have an inclusion $C_{u0}/x_0 \cong I \rightarrow C_u X / X$ and the quotient space is precisely the reduced suspension $\Sigma X$. Since $I$ is a closed contractible subspace of $C_u X / X$ we have that $\text{Vect}_C(C_u X / X) \cong \text{Vect}_C((C_u X / X)/I)$. Hence, $K(C_u X / X) \cong K(\Sigma X)$ and $K(C_u X, X) = \widetilde{K}(C_u X / X) \cong \widetilde{K}(\Sigma X)$.

5.2.3. For a compact pair $(X, A)$ we define $X \cup CA$ to be the space obtained by identifying $A \subseteq X$ with $A \times \{1\}$ in CA. There is a natural homeomorphism $X \cup CA / X \cong CA / A$. Thus, if $A$ is a pointed space we have that $K(X \cup CA, X) = \widetilde{K}(CA, A) \cong \widetilde{K}(\Sigma A) = \widetilde{K}^{-1}(A)$.

5.3. Exact sequences of $K$-groups

We want to relate the $K$-groups of a pair $(X, A)$ with the $K$-groups of $X$ and $A$. We are going to need the following result about “collapsing” vector bundles that we recall from a previous lecture.

**Lemma 5.3.1.** If $A \subseteq X$ is a closed subspace, then any trivialization $\alpha: E|A \cong \tau_n$ on $A$ of a vector bundle $E \rightarrow X$ defines a vector bundle $E/\alpha \rightarrow X/A$ on the quotient $X/A$. \hfill $\square$

**Lemma 5.3.2.** Let $(X, A)$ be a compact pair in $\text{Top}^2$ and let $i: A \rightarrow X$ and $j: (X, \emptyset) \rightarrow (X, A)$ be the canonical inclusions. Then there is an exact sequence

$$K^0(X, A) \xrightarrow{j^*} K^0(X) \xrightarrow{i^*} K^0(A).$$

**Proof.** The composition $(A, \emptyset) \xrightarrow{i} (X, \emptyset) \xrightarrow{j} (X, A)$ factors through $(A, A)$. Applying $K^0$ yields a commutative diagram

$$\begin{align*}
K^0(X, A) \xrightarrow{i^*j^*} K^0(A) \\
K^0(A, A) = \widetilde{K}^0(A/A) = 0.
\end{align*}$$

So, $i^*j^* = 0$ and hence $\text{Im } j^* \subseteq \ker i^*$.

To prove the converse, let $\xi$ be any element in $\ker i^*$. We can represent $\xi$ as a difference $[E] - [\tau_n]$, where $E$ is a vector bundle over $X$ and $\tau_n$ is the trivial bundle of rank $n$ over $X$. By assumption $i^*(\xi) = 0$, which means that $i^*(\xi) = [E|A] - [\tau_n] = 0$. 


So, \([E] = [\tau_n]\) in \(K^0(A)\). This means that these two bundles become isomorphic after we sum with a trivial bundle of certain dimension. More precisely, there is an \(m \geq 0\) such that

\[
\alpha: (E \oplus \tau_m)|A \cong \tau_n \oplus \tau_m.
\]

So, we have found a vector bundle that is trivial in \(A\). By Lemma 5.3.1 we have a vector bundle \((E \oplus \tau_m)/\alpha\) over \(X/A\). Take now \(\eta = [(E \oplus \tau_m)/\alpha] - [\tau_n \oplus \tau_m]\) and observe that \(\eta\) lies in \(\tilde{K}^0(X/A)\) since the rank of \((E \oplus \tau_m)/\alpha\) in the component of the basepoint is \(n + m\). Finally,

\[
j^*(\eta) = [E \oplus \tau_m] - [\tau_n \oplus \tau_m] = [E] - [\tau_n] = \xi,
\]

so \(\ker i^* \subseteq \Im j^*\).

\[\text{Corollary 5.3.3. Let } (X, A) \text{ be a compact pair in } \text{Top}^2 \text{ and } A \text{ in } \text{Top}_\ast. \text{ Then, there is an exact sequence}
\]

\[
K^0(X, A) \xrightarrow{j^*} \tilde{K}^0(X) \xrightarrow{i^*} \tilde{K}^0(A).
\]

\[\text{Proof. We have natural isomorphisms } K^0(X) \cong \tilde{K}^0(X) \oplus K^0(\ast) \text{ and } K^0(A) \cong \tilde{K}^0(A) \oplus K^0(\ast) \text{ and thus the following commutative diagram}
\]

\[
\begin{array}{ccc}
\tilde{K}^0(X) & \xrightarrow{\sim} & \tilde{K}^0(A) \\
\downarrow & & \downarrow \\
K^0(X, A) & \xrightarrow{j^*} & \tilde{K}^0(X) \oplus K^0(\ast) \xrightarrow{i^*} \tilde{K}^0(A) \oplus K^0(\ast) \\
\end{array}
\]

where the central row and the columns are exact. Now, any element in \(K^0(X, A)\) goes to zero in \(K^0(\ast)\) so there is a map \(K^0(X, A) \to \tilde{K}^0(X)\) that makes the diagram commutative. From this it is straightforward to check that the required sequence is exact. \(\Box\)

\[\text{Proposition 5.3.4. Let } (X, A) \text{ be a compact pair of spaces and } A \text{ in } \text{Top}_\ast. \text{ Then there is a natural exact sequence of five terms}
\]

\[
\tilde{K}^{-1}(X) \xrightarrow{i^*} \tilde{K}^{-1}(A) \xrightarrow{\delta} K^0(X, A) \xrightarrow{j^*} \tilde{K}^0(X) \xrightarrow{i^*} \tilde{K}^0(A).
\]

\[\text{Proof. We need to check exactness of the three subsequences of three terms. Exactness of } K^0(X, A) \to \tilde{K}^0(X) \to \tilde{K}^0(A) \text{ is given by Corollary 5.3.3.}
\]

To prove exactness at \(\tilde{K}^{-1}(A) \to K^0(X, A) \to \tilde{K}^0(X)\) we consider the pair of spaces \((X \cup CA, X)\). Applying Corollary 5.3.3 we get an exact sequence

\[
K^0(X \cup CA, X) \xrightarrow{m^*} \tilde{K}^0(X \cup CA) \xrightarrow{k^*} \tilde{K}^0(X)
\]

\[
\theta \downarrow \cong \quad p^* \cong \quad j^* \downarrow
\]

\[
\tilde{K}^{-1}(A) \xrightarrow{\delta} \tilde{K}^0(X/A).
\]
Since $CA$ is contractible, the quotient map $p : X \cup CA \to X/A$ induces isomorphism on $\tilde{K}^0$ and moreover $k^*p^* = j^*$, which follows directly from the commutativity of the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{k} & X \cup CA \\
\downarrow{j} & & \downarrow{p} \\
X \cup CA/CA & \cong & X/A.
\end{array}
$$

We define the connecting homomorphisms $\delta = (p^*)^{-1}m^*\theta^{-1}$, where the morphism $\theta : K^0(X \cup CA, X) \to \tilde{K}^{-1}(A)$ is the isomorphism described in 5.2.3.

Exactness at $\tilde{K}^{-1}(X) \to \tilde{K}^{-1}(A) \to K^0(X, A)$ is a bit more involved. First, we apply Corollary 5.3.3 to the pair $(X \cup C_1A \cup C_2X, X \cup C_1A)$, where we used the notation $C_1$ and $C_2$ to distinguish between the two cones. This gives an exact sequence

$$
\begin{array}{ccc}
K^0(X \cup C_1A \cup C_2X, X \cup C_1A) & \longrightarrow & \tilde{K}^0(X \cup C_1A \cup C_2X) \\
\downarrow{\cong} & & \downarrow{\cong} \\
\tilde{K}^0(X \cup C_1A \cup C_2X/X \cup C_1A) & \cong & \tilde{K}^0(X/A)
\end{array}
$$

By using the definition of $\delta$ given in the previous step, we can check that the composition in the square on the right is indeed $\delta$, as required. For the left part of the diagram we have a square as follows

$$
(5.3.1) \quad \begin{array}{ccc}
K^0(X \cup C_1A \cup C_2X, X \cup C_1A) & \longrightarrow & \tilde{K}^0(X \cup C_1A \cup C_2X) \\
\downarrow{\cong} & & \downarrow{\cong} \\
\tilde{K}^0(X \cup C_1A \cup C_2X/X \cup C_1A) & \cong & \tilde{K}^0(C_1A/A)
\end{array}
$$

Now, we would like the dotted arrow that makes the diagram commutative to be $i^*$ to conclude the proof. We can see that this is not going to be the case since at the level of spaces the diagram

$$
\begin{array}{ccc}
X \cup C_1A \cup C_2X & \xrightarrow{\text{collapse } X \cup C_1A} & C_2X/X \\
\downarrow{\text{collapse } C_2X} & & \downarrow{\text{collapse } C_2X/X} \\
C_1A/A & \xrightarrow{1 \wedge i} & \Sigma A
\end{array}
$$

does not commute. We could try to replace $1 \wedge i$ by $(1 \wedge i) \circ T$, where $T : \Sigma A \to \Sigma A$ is the map that sends $(a, t)$ to $(a, 1-t)$ but the diagram would not commute either.
However, the diagram with \((1 \land i) \circ T\) commutes \textit{up to homotopy}, which is enough for our purposes, since then it will \textit{strictly commute} when we take \(\tilde{K}^0\). So we have the following diagram

\begin{center}
\begin{tikzpicture}

\node (X) at (0,0) {$X \cup C_1 A \cup C_2 X$};
\node (A) at (3,0) {$C_1 A/A$};
\node (B) at (6,0) {$\Sigma A$};
\node (C) at (0,-3) {$C_2 X/X$};
\node (D) at (3,-3) {$C_1 A \cup C_2 A$};
\node (E) at (6,-3) {$\Sigma A$};
\node (F) at (0,-6) {$\Sigma X$};

\draw[->] (X) to node[above] {collapse \(X \cup C_1 A\)} (A);
\draw[->] (A) to node[above] {collapse \(C_2 A\)} (B);
\draw[->] (X) to node[left] {\(\land i\)} (C);
\draw[->] (C) to node[above] {collapse \(X \cup C_1 A\)} (D);
\draw[->] (D) to node[above] {collapse \(C_2 A\)} (E);
\draw[->] (E) to node[left] {\(\land i\)} (F);

\end{tikzpicture}
\end{center}

which induces the following \textit{commutative} diagram after applying \(\tilde{K}^0\)

\begin{center}
\begin{tikzpicture}

\node (X) at (0,0) {$\tilde{K}^0(X \cup C_1 A \cup C_2 X)$};
\node (A) at (3,0) {$\tilde{K}^0(C_1 A/A)$};
\node (B) at (6,0) {$\tilde{K}^{-1}(A)$};
\node (C) at (0,-3) {$\tilde{K}^0(C_2 X/X)$};
\node (D) at (3,-3) {$\tilde{K}^0(C_1 A \cup C_2 A)$};
\node (E) at (6,-3) {$\tilde{K}^{-1}(A)$};
\node (F) at (0,-6) {$\tilde{K}^{-1}(X)$};

\draw[->] (X) to node[above] {\(\cong\)} (A);
\draw[->] (A) to node[above] {\(\cong\)} (B);
\draw[->] (C) to node[above] {\(\cong\)} (D);
\draw[->] (D) to node[above] {\(\cong\)} (E);
\draw[->] (E) to node[above] {\(\cong\)} (F);
\draw[->] (X) to node[left] {\(\cong\)} (C);
\draw[->] (C) to node[left] {\(\cong\)} (D);
\draw[->] (D) to node[left] {\(\cong\)} (E);
\draw[->] (E) to node[left] {\(\cong\)} (F);

\end{tikzpicture}
\end{center}

By “inserting” this diagram into diagram (5.3.1), we can check that the latter commutes if the dotted arrow is \(T^* \circ i^*\). In the exercises we will prove that the map \(T^*: \tilde{K}^{-1}(A) \to \tilde{K}^{-1}(A)\) sends every element to its inverse. So in the end, we get an exact sequence

\[ \tilde{K}^{-1}(X) \xrightarrow{-i^*} \tilde{K}^{-1}(A) \xrightarrow{\delta} K^0(X, A). \]

But since \(-i^*\) and \(i^*\) have both the same kernel and the same image, we can replace \(-i^*\) by \(i^*\) and we still have an exact sequence. This completes the proof. \qed

\textbf{Corollary 5.3.5.} Let \((X, A)\) be a compact pair and \(A\) in \(\text{Top}_+\). Then there is a long exact sequence

\[ \cdots \to \tilde{K}^{-2}(X) \xrightarrow{i^*} \tilde{K}^{-2}(A) \xrightarrow{\delta} K^{-1}(X, A) \xrightarrow{j^*} \tilde{K}^{-1}(X) \xrightarrow{i^*} \]

\[ \xrightarrow{i^*} \tilde{K}^{-1}(A) \xrightarrow{\delta} K^0(X, A) \xrightarrow{j^*} \tilde{K}^0(X) \xrightarrow{i^*} \tilde{K}^0(A). \]

\textbf{Proof.} Replace in the exact sequence of Proposition 5.3.4 the compact pair \((X, A)\) by \((\Sigma^n X, \Sigma^n A)\) for \(n = 1, 2, \ldots\) \qed

\textbf{Corollary 5.3.6.} Let \((X, A)\) be a compact pair. Then there is a long exact sequence

\[ \cdots \to K^{-2}(X) \xrightarrow{i^*} K^{-2}(A) \xrightarrow{\delta} K^{-1}(X, A) \xrightarrow{j^*} K^{-1}(X) \xrightarrow{i^*} \]

\[ \xrightarrow{i^*} K^{-1}(A) \xrightarrow{\delta} K^0(X, A) \xrightarrow{j^*} K^0(X) \xrightarrow{i^*} K^0(A). \]

\textbf{Proof.} The result follows directly by applying Corollary 5.3.5 to the pair \((X_+, A_+)\) and using that \(\tilde{K}^{-n}(X_+) = K^{-n}(X)\). \qed
LECTURE 6: $K$-THEORY AS A COHOMOLOGY THEORY

In this lecture we will prove that $K$-theory is a generalized cohomology theory. For this we will need to use the Bott periodicity theorem in order to define the positive $K$-groups. A detailed proof of Bott periodicity will be given in the third part of the course.

6.1. Eilenberg–Steenrod axioms for cohomology

Eilenberg and Steenrod introduced in 1945 an axiomatic approach to cohomology (and homology) theory by abstracting the fundamental properties that any cohomology theory should satisfy.

6.1.1. A cohomology theory $h^*$ on $\text{Top}^2$ (or any nice subcategory like compact pairs, pairs of CW-complexes, ...) is a collection of contravariant functors

$$h^n: \text{Top}^2 \rightarrow \text{Ab}, \ n \in \mathbb{Z},$$

where $\text{Ab}$ denotes the category of abelian groups, and natural transformations

$$\delta^n: h^n \circ R \rightarrow h^{n+1},$$

where $R: \text{Top}^2 \rightarrow \text{Top}^2$ is the functor that sends $(X, A)$ to $(A, \emptyset)$ and $f$ to $f|A$, satisfying the following axioms:

(i) Homotopy invariance. If $f \simeq g$, then $h^n(f) = h^n(g)$ for every $n \in \mathbb{Z}$.

(ii) Excision. For every pair $(X, A)$ and $U \subseteq A$ such that the closure $\overline{U}$ is contained in the interior $A^o$, the inclusion $(X \setminus U, A \setminus U) \rightarrow (X, A)$ induces an isomorphism

$$h^n(X \setminus U, A \setminus U) \cong h^n(X, A), \text{ for every } n \in \mathbb{Z}.$$

(iii) Exactness. For every pair $(X, A)$, consider the inclusions $i: A \rightarrow X$ and $j: (X, \emptyset) \rightarrow (X, A)$. Then there is a long exact sequence

$$\cdots \rightarrow h^{n-1}(A) \xrightarrow{\delta^{n-1}} h^n(X, A) \xrightarrow{j^*} h^n(X) \xrightarrow{i^*} h^n(A) \xrightarrow{\delta^n} h^{n+1}(X, A) \xrightarrow{j^*} h^{n+1}(X) \xrightarrow{i^*} h^{n+1}(A) \rightarrow \cdots$$

If moreover $h^*$ satisfies the dimension axiom, that is, $h^n(*) = 0$ for every $n \in \mathbb{Z}$, then $h^*$ is called an ordinary cohomology theory; otherwise it is called an generalized or extraordinary cohomology theory.

For example, singular, cellular, de Rham and Čech cohomology are all ordinary cohomology theories. They all coincide on finite CW-pairs. However, $K$-theory will be a generalized cohomology theory.

6.1.2. A reduced cohomology theory $\tilde{h}^*$ on $\text{Top}_*$ is a collection of contravariant functors

$$\tilde{h}^n: \text{Top}_* \rightarrow \text{Ab}, \ n \in \mathbb{Z},$$

and natural equivalences $\tilde{h}^n \circ \Sigma \cong \tilde{h}^{n+1}$ satisfying the following axioms:

(i) Homotopy invariance. If $f \simeq g$, then $\tilde{h}^n(f) = \tilde{h}^n(g)$ for every $n \in \mathbb{Z}$.
(ii) **Exactness.** For every pair \((X, A)\) in \(\text{Top}^2\) and \(A\) in \(\text{Top}_*\) there is an exact sequence
\[
\tilde{h}^n(X \cup CA) \xrightarrow{j^*} \tilde{h}^n(X) \xrightarrow{i^*} \tilde{h}^n(A),
\]
for every \(n \in \mathbb{Z}\),

where \(i: A \to X\) and \(j: X \to X \cup CA\) denote the inclusions.

**Theorem 6.1.3.** \(K\)-theory and reduced \(K\)-theory are a generalized cohomology theory and a reduced cohomology theory, respectively.

**Proof.** We have already defined the negative \(K\)-groups (see Definition 5.2.1). To define the positive ones, we have to use the Bott periodicity theorem. This theorem states that there is an isomorphism
\[
\beta: \tilde{K}^{-n}(X) \cong \tilde{K}^{-n-2}(X), \text{ for all } n \geq 0.
\]

Since, for a space \(X\) in \(\text{Top}\) we have that \(K^{-n}(X) = \tilde{K}^{-n}(X_+)\) there is also an isomorphism \(K^{-n}(X) \cong K^{-n-2}(X)\) in the unreduced case.

Thus, for a space \(X\) in \(\text{Top}_*\), we can define
\[
\tilde{K}_{2n}(X) = \tilde{K}^0(X) \text{ and } \tilde{K}_{2n+1}(X) = \tilde{K}^{-1}(X) \text{ for every } n \in \mathbb{Z}.
\]

And similarly, for a space \(X\) in \(\text{Top}\), we define
\[
K_{2n}(X) = K^0(X) \text{ and } K_{2n+1}(X) = K^{-1}(X) \text{ for every } n \in \mathbb{Z}.
\]

This allows to extend all the results about exact sequences from the previous lectures to all the integers. In particular we can extend the long exact sequence of Corollary 5.3.6 to an infinite long exact sequence also on the right.

Homotopy invariance for \(K^*\) and \(\tilde{K}^*\) follows from the fact that if we have a vector bundle and we pullback along homotopic maps, then we get isomorphic bundles, and is left as an exercise.

Exactness for \(K^*\) is precisely Corollary 5.3.6 and for \(\tilde{K}^*\) it follows from Corollary 5.3.3.

The excision axiom is essentially the fact that \(K^n(X, A) = \tilde{K}(X/A)\). To prove it, let \(X = X_1 \cup X_2\) and also \(X = X_1' \cup X_2'\). Then \(X_1/X_1' \cap X_2 \cong X/X_2\) and thus
\[
K^n(X_1, X_1 \cap X_2) = \tilde{K}^n(X_1/X_1' \cap X_2) \cong \tilde{K}^n(X/X_2) = \tilde{K}^n(X, X_2).
\]

Now, let \(X_1 = X \setminus U\) and \(X_2 = A\) and observe that
\[
(X \setminus U)^o \cup A^o = (X \setminus U) \cup A^o \supseteq (X \setminus A^o) \cup A^o = X.
\]

So, we can apply the previous result to get excision. \(\square\)

**Remark 6.1.4.** Due to Bott periodicity we have only two different \(K\)-groups \(K^0\) and \(K^1\). So, alternatively, the long exact sequence of Corollary 5.3.6 can be written as an exact sequence of six terms
\[
\begin{array}{ccc}
K^0(X, A) & \longrightarrow & K^0(X) \longrightarrow K^0(A) \\
& & \downarrow \\
K^1(A) & \leftarrow & K^1(X) \longrightarrow K^1(X, A).
\end{array}
\]

**Corollary 6.1.5.** Let \(X\) and \(Y\) in \(\text{Top}_*\). Then \(\tilde{K}^{-n}(X \lor Y) \cong \tilde{K}^{-n}(X) \oplus \tilde{K}^{-n}(Y)\) for every \(n \geq 0\).
Proof. We have (pointed) inclusions \( i_1: X \to X \vee Y \) and \( i_2: Y \to X \vee Y \) and surjections \( r_1: X \vee Y \to X \) and \( r_2: Y \to X \vee Y \). They satisfy that \( r_1i_1 = \text{id}_X \) and \( r_2i_2 = \text{id}_Y \). So taking \( \widetilde{K}^{-n} \) we have maps

\[
\widetilde{K}^{-n}(X) \oplus \widetilde{K}^{-n}(Y) \xrightarrow{r_1^* + r_2^*} \widetilde{K}^{-n}(X \vee Y) \xrightarrow{(i_1^*, i_2^*)} \widetilde{K}^{-n}(X) \oplus \widetilde{K}^{-n}(Y)
\]

and \((i_1^*, i_2^*) \circ (r_1^* + r_2^*) = \text{id} \). This means that \((i_1^*, i_2^*)\) is surjective.

To prove that it is also injective, let \( \xi \in \ker(i_1^*, i_2^*) \). Then \( i_1^*(\xi) = 0 \) and \( i_2^*(\xi) = 0 \). Consider the pair \((X \vee Y, X)\) and apply Corollary 5.3.5 to get an exact sequence

\[
\widetilde{K}^{-n}(X \vee Y, X) = \widetilde{K}^{-n}(X \vee Y/X) \cong \widetilde{K}^{-n}(Y) \xrightarrow{r_2^*} \widetilde{K}^{-n}(X \vee Y) \xrightarrow{i_1^*} \widetilde{K}^{-n}(X).
\]

Since \( \xi \in \ker i_1^* \), there exists an element \( \eta \in \widetilde{K}^{-n}(Y) \) such that \( r_2^*(\eta) = \xi \). But \( \eta = i_2^*r_2^*(\eta) = i_2^*(\xi) = 0 \). Hence \( \eta = 0 \) and so \( \xi = 0 \) too. \( \square \)

Corollary 6.1.6. Let \((X, A)\) in \( \text{Top}^2 \) and \( A \) in \( \text{Top}_+ \). If \( A \) is contractible, then \( \widetilde{K}^{-n}(X/A) \cong \widetilde{K}^{-n}(X) \) for every \( n \geq 0 \).

Proof. Use the long exact sequence of Corollary 5.3.5 and the fact that if \( A \) is contractible, then \( \Sigma^n A \) is also contractible, hence \( \widetilde{K}^{-n}(A) = 0 \) for every \( n \geq 0 \). \( \square \)

Corollary 6.1.7. Let \( X \) and \( Y \) in \( \text{Top}_+ \) and \( Y \) be a retract of \( X \). Then

\[
\widetilde{K}^{-n}(X) \cong \widetilde{K}^{-n}(X,Y) \oplus \widetilde{K}^{-n}(Y)
\]

for every \( n \geq 0 \).

Proof. Since \( Y \) is a retract of \( X \), there exists a map \( r: X \to Y \) such that \( ri = \text{id}_Y \), where \( i \) denotes the inclusion. This means that \( i^*r^* = \text{id} \) and hence that \( i^* \) is injective. Thus, in the long exact sequence of Corollary 5.3.5 the map \( \delta \) factors through the zero map and therefore we have split short exact sequences

\[
0 \to \widetilde{K}^{-n}(X,Y) \xrightarrow{\delta} \widetilde{K}^{-n}(X) \xrightarrow{i_2^*} \widetilde{K}^{-n}(Y) \to 0,
\]

since \( r^* \) is a section. So \( \widetilde{K}^{-n}(X) \cong \widetilde{K}^{-n}(X,Y) \oplus \widetilde{K}^{-n}(Y) \). \( \square \)

Remark 6.1.8. The same result is true in the unreduced case for \( X \) and \( Y \) in \( \text{Top} \) and \( Y \) a retract of \( X \). Namely,

\[
\widetilde{K}^{-n}(X) \cong \widetilde{K}^{-n}(X,Y) \oplus \widetilde{K}^{-n}(Y)
\]

for every \( n \geq 0 \). It can be deduced from the previous case by replacing \( X \) and \( Y \) by \( X_+ \) and \( Y_+ \), respectively, and using that \( \widetilde{K}^{-n}(X_+) = \widetilde{K}^{-n}(X) \).

Corollary 6.1.9. Let \( X \) and \( Y \) in \( \text{Top}_+ \). Then the projection maps \( \pi_1: X \times Y \to X \), \( \pi_2: X \times Y \to Y \) and the quotient map \( q: X \times Y \to X \times Y/X \vee Y = X \wedge Y \) induce an isomorphism

\[
\widetilde{K}^{-n}(X \times Y) \cong \widetilde{K}^{-n}(X \wedge Y) \oplus \widetilde{K}^{-n}(X) \oplus \widetilde{K}^{-n}(Y)
\]

for every \( n \geq 0 \).

Proof. Using the map \( X \to X \times Y \) that sends \( x \) to \((x, y_0)\) and the projection \( \pi_1 \) we can see that \( X \) is a retract of \( X \times Y \). By Corollary 6.1.7 we have that

\[
\widetilde{K}^{-n}(X \times Y) \cong \widetilde{K}^{-n}(X \times Y, X) \oplus \widetilde{K}^{-n}(X).
\]
Now, $K^{-n}(X \times Y, X) = \widetilde{K}^{-n}(X \times Y/X)$. But $Y$ is a retract of $X \times Y/X$, so applying Corollary 6.1.7 again, we obtain

$$
\widetilde{K}^{-n}(X \times Y/X) \cong K^{-n}(X \times Y/X, Y) \oplus \widetilde{K}^{-n}(Y).
$$

But $K^{-n}(X \times Y/X, Y) = \widetilde{K}^{-n}(X \vee Y)$ yielding the result. \hfill \Box

Remark 6.1.10. If we assume Bott periodicity, then all of the previous corollaries hold for $K^n$ and $\widetilde{K}^n$ for $n \in \mathbb{Z}$.

### 6.2. The external product for reduced $K$-theory

#### 6.2.1. Let $X$ and $Y$ in $\text{Top}_*$ and consider the external product

\[
K^0(X) \otimes K^0(Y) \rightarrow K^0(X \times Y)
\]

$$
\xi \otimes \xi' \mapsto \xi \ast \xi' = \pi_1^\ast(\xi) \cdot \pi_2^\ast(\xi'),
$$

where $\pi_1$ and $\pi_2$ denote the projections. This external product is induced by the tensor product of vector bundles

$$
\text{Vec}_C(X) \times \text{Vec}_C(Y) \rightarrow \text{Vec}_C(X \times Y)
\]

$$
E \otimes F \mapsto \pi_1^\ast(E) \otimes \pi_2^\ast(F).
$$

#### 6.2.2. Now, let us see what happens if we restrict this external product to elements in $\tilde{K}^0(X) \otimes \tilde{K}^0(Y)$. Recall that $\tilde{K}^0(X) = \ker(K^0(X) \rightarrow K^0(x_0))$. Let $\xi \in \tilde{K}^0(X)$ and $\xi' \in \tilde{K}^0(Y)$. Then from the following commutative diagram

\[
\begin{array}{ccc}
K^0(\{x_0\} \times Y) & \longrightarrow & K^0(X \times Y) \\
\downarrow & & \downarrow \pi_1^\ast \\
K^0(x_0) & \leftarrow & K^0(X)
\end{array}
\]

it follows that $\pi_1^\ast(\xi)$ that lies in $K^0(X \times Y)$ restricts to zero in $K^0(\{x_0\} \times Y)$. Similarly $\pi_2^\ast(\xi')$ restricts to zero in $K^0(X \times \{y_0\})$. So, $\pi_1^\ast(\xi) \cdot \pi_2^\ast(\xi')$ restrict to zero in $K^0(X \vee Y)$ and hence, it lies in the kernel of $K^0(X \times Y) \rightarrow K^0(*)$ which is $\tilde{K}^0(X \times Y)$. By Corollary 6.1.9 there is a split short exact sequence

$$
0 \rightarrow \tilde{K}^0(X \wedge Y) \rightarrow \tilde{K}^0(X \times Y) \rightarrow \tilde{K}^0(X) \oplus \tilde{K}^0(Y) \cong \tilde{K}^0(X \vee Y) \rightarrow 0.
$$

Since $\pi_1^\ast(\xi) \cdot \pi_2^\ast(\xi')$ lies in $\tilde{K}^0(X \times Y)$ and is zero in $\tilde{K}^0(X \times Y)$ it lies in the kernel of the third map in the above sequence, which is $\tilde{K}^0(X \wedge Y)$. So we have defined a map

$$
\tilde{K}^0(X) \otimes \tilde{K}^0(Y) \rightarrow \tilde{K}^0(X \wedge Y).
$$

This map is, in fact the restriction of the exterior product on $K^0$ as we can see in the following diagram

\[
\begin{array}{ccc}
K^0(X) \otimes K^0(Y) & \cong & \tilde{K}^0(X) \otimes \tilde{K}^0(Y) \oplus \tilde{K}^0(X) \oplus \tilde{K}^0(Y) \oplus \mathbb{Z} \\
K^0(X \times Y) & \cong & \tilde{K}^0(X \wedge Y) \oplus \tilde{K}^0(X) \oplus \tilde{K}^0(Y) \oplus \mathbb{Z},
\end{array}
\]

where the first isomorphism is obtained by using that $K^0(X) \cong \tilde{K}^0(X) \oplus \mathbb{Z}$ and similarly for $Y$, and the isomorphism on the second row is obtained by using Corollary 6.1.9.
6.2.3. We can replace \( X \) by \( \Sigma^n X \) and \( Y \) by \( \Sigma^m Y \) in (1) to obtain a pairing
\[
\tilde{K}^{-n}(X) \otimes \tilde{K}^{-m}(Y) \to \tilde{K}^{-n-m}(X \wedge Y).
\]
If \( X \) and \( Y \) are in \( \text{Top} \), we can replace \( X \) by \( X_+ \) and \( Y \) by \( Y_+ \) in the previous pairing to obtain a pairing
\[
K^{-n}(X) \otimes K^{-m}(Y) \to K^{-n-m}(X \wedge Y),
\]
in the unreduced case.

6.3. Vector bundles on spheres and clutching functions

The only computation we have so far for the \( K \)-groups is \( K^0(X) \cong \mathbb{Z} \) if \( X \) is a contractible space. The next natural step is to consider spheres.

6.3.1. We can decompose the sphere \( S^k \) as the union of the upper and the lower hemisphere. Since each hemisphere is contractible (it is homotopy equivalent to a disk) every fiber bundle on \( S^k \) restricts to a trivial bundle on each of the hemispheres. So a fiber bundle on \( S^k \) “should be determined” by a map from the intersection of the two hemispheres to \( GL_n(\mathbb{C}) \).

Definition 6.3.2. A clutching function for \( S^k \) is a map \( f : S^{k-1} \to GL_n(\mathbb{C}) \), where \( GL_n(\mathbb{C}) \) is the group of invertible \( n \times n \) matrices with coefficients in \( \mathbb{C} \).

6.3.3. Every clutching function \( f : S^{k-1} \to GL_n(\mathbb{C}) \) gives rise to a vector bundle \( E_f \) over \( S^k \) of rank \( n \). We define
\[
E_f = (D^- \times \mathbb{C}^n) \cup_{S^{k-1} \times \mathbb{C}^n} (D^+ \times \mathbb{C}^n),
\]
where \( D^- = \{(x_1, \ldots, x_{k+1}) \in S^k \mid x_{k+1} \leq 0 \} \) is the lower hemisphere, and similarly, \( D^+ = \{(x_1, \ldots, x_{k+1}) \in S^k \mid x_{k+1} \geq 0 \} \) is the upper hemisphere. If \( x \in S^{k-1} \), then we identify \((x,v)\) in \( D^- \times \mathbb{C}^n \) with \((x,f(x)v)\) in \( D^+ \times \mathbb{C}^n \). If \( f \) is homotopic to \( g \), then \( E_f \cong E_g \).

In fact, as we saw in the exercises, there is a bijection between homotopy classes of clutching functions from a space \( X \) to \( GL_n(\mathbb{C}) \) and \( \text{Vect}_c^m(\Sigma X) \). In the case of spheres, this particularizes to the following result.

Proposition 6.3.4. There is an isomorphism \( \text{Vect}_c^m(S^k) \cong [S^{k-1}, GL_n(\mathbb{C})] \) for every \( n, k \geq 1 \). \( \square \)

Lemma 6.3.5. The group \( GL_n(\mathbb{C}) \) is path-connected for every \( n \geq 1 \).

Proof. The case \( n = 1 \) is trivial since \( GL_1(\mathbb{C}) = \mathbb{C} \setminus \{0\} \cong \mathbb{R}^2 \setminus \{0\} \), which is path-connected. Let \( n \geq 2 \) and let \( A \in GL_n(\mathbb{C}) \). We are going to show that \( A \) is connected to the identity matrix by a path. Let \( B \) be the Jordan canonical form of \( A \), that is,
\[
B = \begin{pmatrix}
J_1 & & \\
& \ddots & \\
& & J_k
\end{pmatrix}, \text{ where } J_i = \begin{pmatrix}
\lambda_i & 1 & 1 \\
& \ddots & 1 \\
0 & & \lambda_i
\end{pmatrix},
\]
and there exists and invertible matrix \( C \) such that \( A = C B C^{-1} \).

For each \( \lambda_i \in \mathbb{C} \), let \( \gamma_i : I \to \mathbb{C} \) be a path from \( \lambda_i \) to \( 1 \) not passing through the origin. Let \( B(t) \) be matrix obtained from \( B \) by replacing \( \lambda_i \) by \( \gamma_i \) and multiplying by \((1-t)\) all the elements above the diagonal.
Now define the path $\gamma: I \to GL_n(\mathbb{C})$ by $\gamma(t) = CB(t)C^{-1}$. This path satisfies that $\gamma(0) = CBC^{-1} = A$ and $\gamma(1) = CC^{-1} = \text{Id}$. □

**Corollary 6.3.6.** Every complex vector bundle over $S^1$ is trivial. In particular, $K^0(S^1) \cong \mathbb{Z}$.  

**Proof.** By Proposition 6.3.4 we know that $\text{Vect}^n(S^1) \cong [S^0, GL_n(\mathbb{C})]$ consists of one element only, since $GL_n(\mathbb{C})$ is path-connected by Lemma 6.3.5. □

**Corollary 6.3.7.** Every complex line bundle over $S^k$ for $k > 2$ is trivial.  

**Proof.** By Proposition 6.3.4 we know that $\text{Vect}^1(S^k) \cong [S^{k-1}, GL_1(\mathbb{C})] \cong [S^{k-1}, S^1]$, since $GL_1(\mathbb{C}) \simeq U(1) = S^1$, where $U(1)$ is the unitary group of order one (in fact, $GL_n(\mathbb{C}) \simeq U(n)$ for every $n$).

The sphere $S^{k-1}$ is simply connected for $k > 2$, hence any map $S^{k-1} \to S^1$ factors through the universal cover $\mathbb{R} \to S^1$. Since $\mathbb{R}$ is contractible, any map $S^{k-1} \to S^1$ is homotopic to a constant map. But any two constant maps on $S^1$ are homotopic because $S^1$ is path-connected. So $[S^{k-1}, S^1]$ has one element only. □

**Corollary 6.3.8.** As abelian group under the tensor product $\text{Vect}^2(S^2) \cong \mathbb{Z}$.  

**Proof.** By Proposition 6.3.4 we know that $\text{Vect}^2(S^2) \cong [S^1, S^1] \cong \mathbb{Z}$, since there is an isomorphism $\pi_1(S^1, x) \cong [S^1, S^1]$. □
LECTURE 7: K-THEORY GROUPS OF THE SPHERES

In the previous lecture we proved that \( K^0(S^1) \cong \mathbb{Z} \). The aim of this lecture is to compute the \( K \)-theory groups of all spheres and to state in a precise way the Bott periodicity theorem, that we used to prove that \( K \)-theory is a generalized cohomology theory.

7.1. Clutching functions for vector bundles over \( S^2 \)

7.1.1. Recall that vector bundles over \( S^k \) are determined by clutching functions \( S^{k-1} \to GL_n(\mathbb{C}) \). Thus, for vector bundles over the sphere \( S^2 \), we have to study functions \( S^1 \to GL_n(\mathbb{C}) \).

Recall also that for \( \mathbb{C}P^1 = S^2 \) we have the canonical line bundle, that we denote by \( H \),

\[
\{ (\ell, v) \mid \ell \in \mathbb{C}P^1, \ v \in \ell \} \to \mathbb{C}P^1 = S^2
\]

that sends \((\ell, v)\) to \(\ell\). This vector bundle has clutching function \( f: S^1 \to GL_n(\mathbb{C}) \) defined by \( f(z) = z \), that is, \( f(z)(v) = zv \) for every \( z \in S^1 \) and \( v \in \mathbb{C} \).

The clutching function of the sum of two vector bundles is the block sum of the clutching functions of the two bundles. Thus, the bundle \( H \oplus H \) has clutching function given by

\[
f(z) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}.
\]

The tensor product \( H \otimes H \) has clutching function \( z^2 \) and so \( (H \otimes H) \oplus \tau_1 \) has clutching function

\[
g(z) = \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}.
\]

7.1.2. Although it does not seem so, the clutching functions \( f \) and \( g \) give rise to isomorphic bundles over \( S^2 \). Indeed, we can construct an explicit homotopy between \( g \) and \( f \) as follows. Let \( H: S^1 \times [0, 1] \to GL_2(\mathbb{C}) \) be the map that sends \((z, t)\) to the product

\[
\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \pi t/2 & -\sin \pi t/2 \\ \sin \pi t/2 & \cos \pi t/2 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \pi t/2 & \sin \pi t/2 \\ -\sin \pi t/2 & \cos \pi t/2 \end{pmatrix}.
\]

For every \( z \in S^1 \) and \( t \in [0, 1] \) the matrix \( H(z, t) \) is invertible and \( H(z, 0) = g(z) \) and \( H(z, 1) = f(z) \). So, we have proved the following

**Proposition 7.1.3.** Let \( H \) be the canonical line bundle over \( S^2 \). Then we have that \( H \oplus H \cong (H \otimes H) \oplus \tau_1 \) as bundles of rank 2 over \( S^2 \).

7.1.4. Observe that the isomorphism of Proposition 7.1.3 shows that in \( K^0(S^2) \) we have that

\[
([H] - [\tau_1]) \otimes ([H] - [\tau_1]) = [H \otimes H] - [H \oplus H] + [\tau_1] = 0.
\]

This means that we have a well-defined ring homomorphism

\[
\mu: \mathbb{Z}[H]/(H - 1)^2 \longrightarrow K^0(S^2)
\]
that sends $H$ to $[H]$ and 1 to $[τ_1]$ and which allows to formulate the Bott periodicity theorem:

**Theorem 7.1.5** (Bott periodicity theorem — product version). For every space $X$ in $\text{Top}$, the morphism

$$\tilde{μ}: K^0(X) \otimes \mathbb{Z}[H]/(H - 1)^2 \xrightarrow{1\otimesμ} K^0(X) \otimes K^0(S^2) \to K^0(X \times S^2),$$

where the second map is the external product, is an isomorphism.

We will not prove the theorem in its full generality, but will restrict to the proof in the case $X = \ast$, which will give a computation of $K^0(S^2)$.

**Theorem 7.1.6.** The map $μ: \mathbb{Z}[H]/(H - 1)^2 \to K^0(S^2)$ is an isomorphism.

### 7.2. The proof of Theorem 7.1.6

#### 7.2.1. We will divide the proof in several parts. Since every vector bundle is determined by a clutching function we will proceed according to the following steps:

(i) Prove that if $f$ is a linear clutching function, then the associated bundle is isomorphic to a linear combination of $H$ and $τ_1$ (that is, it is in the image of $μ$).

(ii) Extend the previous result to polynomial clutching functions.

(iii) Extend the previous result to Laurent polynomial clutching functions.

(iv) Prove that any clutching function is homotopic to a Laurent polynomial clutching function.

This will show that $μ$ is surjective. And finally:

(v) Prove that if $μ$ is injective.

#### 7.2.2. We will start by considering simple functions of the form $f(z) = \text{Id}z + B$, where $\text{Id}$ denotes the identity matrix. Observe that $f$ is a clutching function if and only if $\det(\text{Id}z + B) \neq 0$ for all $z \in S^1$, or equivalently if and only if $(\text{Id}z + B)(v) \neq 0$ for all $z \in S^1$ and all $v \neq 0$. This happens if and only if $Bv \neq -zv$ for all $z \in S^1$ and all $v \neq 0$, that is, when the matrix $B$ has no eigenvalues in $S^1$.

**Lemma 7.2.3.** Let $f(z) = \text{Id}z + B$ be a clutching function. Then:

(i) The matrix $B$ has all eigenvalues outside $S^1$ if and only if $H(z, t) = \text{Id}tz + B$ is a homotopy of clutching functions between $B$ and $f(z)$.

(ii) The matrix $B$ has all eigenvalues inside $S^1$ if and only if $H(z, t) = \text{Id}z + tB$ is a homotopy of clutching functions between $\text{Id}z$ and $f(z)$.

**Proof.** We prove only part (i). Part (ii) is proved similarly and is left as an exercise. The map $H(z, t)$ is a clutching function for every $t$ if and only if $(\text{Id}tz + B)(v) \neq 0$ for every $t \in [0, 1]$, $z \in S^1$ and $v \neq 0$. This happens if and only if $−tz$ is not an eigenvalue of $B$ for every $t \in [0, 1]$ and $z \in S^1$, that is, if and only if all eigenvalues of $B$ are outside $S^1$. \(\square\)

Thus, if $f(z) = \text{Id}z + B$ is a clutching function and

(i) $B$ has all eigenvalues outside $S^1$, then $f(z) \simeq B$ and the bundle associated to $f$ is isomorphic to $nτ_1$ for some $n \geq 0$ (recall that $GL_n(\mathbb{C})$ is path connected and so $B$ is connected by a path to the identity matrix);

(ii) $B$ has all eigenvalues inside $S^1$, then $f(z) \simeq \text{Id}z$ and the bundle associated to $f$ is isomorphic to $nH$ for some $n \geq 0$. 


7.2.4. In general, $B$ will have eigenvalues inside and outside $S^1$. To deal with this case, we will use the following result that we will not prove.

**Lemma 7.2.5.** Let $B$ be an $n \times n$ matrix with coefficients in $\mathbb{C}$ and no eigenvalues in $S^1$. Then there are subspaces $V_+$ and $V_-$ of $\mathbb{C}^n$ such that

(i) $\mathbb{C} = V_+ \oplus V_-.$

(ii) $V_+$ and $V_-$ are invariant under $B$.

(iii) The restriction $B_+$ of $B$ to $V_+$ has all eigenvalues outside $S^1$ and the restriction $B_-$ of $B$ to $V_-$ has all eigenvalues inside $S^1$. \hfill $\square$

This means that the matrix $B$ is similar (that is, conjugate) to the block matrix

$$
\begin{pmatrix}
    B_+ & 0 \\
    0 & B_-
\end{pmatrix}.
$$

Hence, the matrix $\text{Id} z + B$ is similar to the matrix

$$
\begin{pmatrix}
    \text{Id} z + B_+ & 0 \\
    0 & \text{Id} z + B_-
\end{pmatrix}.
$$

By Lemma 7.2.3, the bundle associated to this last clutching function is isomorphic to $k\tau_1 \oplus (n-k)H$ for some $k \geq 0$. Thus, we have proved

**Lemma 7.2.6.** Any vector bundle over $S^2$ with clutching function $f(z) = \text{Id} z + B$ is a linear combination of $\tau_1$ and $H$. \hfill $\square$

7.2.7. Now, we want to extend the previous lemma to linear clutching functions of the form $f(z) = Az + B$. The idea is to reduce to the case of a function of the form $\text{Id} z + B'$. We could try to multiply $f(z)$ by $A^{-1}$, but $A$ is not invertible in general. However, $f(z)$ will be homotopic to a linear map in which the first coefficient is invertible.

**Lemma 7.2.8.** Any bundle over $S^2$ with clutching function $f(z) = Az + B$ is a linear combination of $\tau_1$ and $H$.

**Proof.** Consider the function $H(z,t) = (A + tB)z + (tA + B)$. We have that $H(z,0) = f(z)$, but $H(z,1)$ is not a clutching function, since $H(-1,1) = 0$. So $H(z,t)$ give a homotopy of clutching functions for $t < 1$. Indeed, to see that $H(z,t)$ is invertible for all $z \in S^1$ and $t < 1$ we write

$$
H(z,t) = A(z + t) + B(1 + tz) = (1 + tz) \left( A \frac{z + t}{1 + tz} + B \right),
$$

and note that $(z + t)/(1 + tz)$ is in $S^1$ for all $z \in S^1$ and $t < 1$, since

$$
\frac{|z + t|}{|1 + tz|} = \frac{|z\overline{z} + t\overline{z}|}{|1 + tz|} = \frac{|1 + tz|}{|1 + tz|} = \frac{|v|}{|v|} = 1.
$$

So, $A(z + t)/(1 + tz) + B$ is invertible for all $z \in S^1$ and $t < 1$ and hence so is $H(z,t)$.

For $z = 1$, we have that $f(1) = A + B$ is invertible. The function $\det(A + tB)$ is continuous, so there is a neighborhood $V$ of 1 such that $\det(A + tB) \neq 0$ for all $t \in V$. Thus, taking $t_0 \in V$, we have that $f(z) = Az + B$ is homotopic to $(A + t_0B)z + (t_0A + B)$. But now, we can divide by $A + t_0B$, so the clutching functions $(A + t_0B)z + (t_0A + B)$ and $\text{Id} z + (t_0A + B)(A + t_0B)^{-1}$ give isomorphic bundles.

In the end, we have proved that $f(z) = Az + B$ is homotopic to $\text{Id} z + B'$ for some matrix $B'$. The result now follows from Lemma 7.2.6. \hfill $\square$
Lemma 7.2.11. Every vector bundle over $K$ function is equivalent in $f$. Next we consider Laurent polynomial clutching functions of the form obtained from $f$. Then, in $K^0(S^2)$, we will have that $[E_f] = [E_{L^n(f)}] - [\tau_m]$.

Lemma 7.2.11. Every vector bundle over $S^2$ with Laurent polynomial clutching function is equivalent in $K^0(S^2)$ to a linear combination of $\tau_1$ and $H$.

Proof. If $f(z)$ is a Laurent polynomial clutching function, then $f(z) = z^{-m}g(z)$ for some $m \geq 0$ and a polynomial clutching function $g(z)$. Then $[E_f] = [E_g \otimes H^{-m}]$.

By 7.2.9, $[E_g] = [E_{L^n(g)}] - [\tau_k]$. To take care of the term $H^{-m}$ we can use the relation $(H - 1)^2 = 0$. An easy induction argument shows that $H^n = nH - (n-1)$ for every $n \in \mathbb{Z}$. □

7.2.10. Next we consider Laurent polynomial clutching functions of the form $f(z) = \sum_{|i|\leq n} A_i z^i$.

Theorem 7.2.13. Let $f : S^1 \to \mathbb{C}$ be a continuous functions. Then for every $\varepsilon > 0$ there is a Laurent polynomial clutching function $g$ such that $|f(z) - g(z)| < \varepsilon$ for all $z \in S^1$. □

Proposition 7.2.14. Let $f : S^1 \to GL_n(\mathbb{C})$ be a clutching function. Then there is a Laurent polynomial clutching function $g : S^1 \to GL_n(\mathbb{C})$ such that $f \simeq g$ and the homotopy is through clutching functions.

Proof. Consider the set of all functions $S^1 \to M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ denotes the $n \times n$ matrices with coefficients in $\mathbb{C}$. This is a vector space over $\mathbb{C}$ and we can define a norm

$$|f| = \sup_{z \in S^1, |v| = 1} |f(z)(v)|.$$

Using this norm we can put a topology on the set of all functions from $S^1$ to $M_n(\mathbb{C})$ and we claim that in this topology the clutching functions from an open subset.

First note that if $f$ is a clutching function with matrix components $f_{ij}$ and $\varepsilon > 0$, then we have by Theorem 7.2.13 that there are Laurent polynomials $g_{ij}$ such that $|f_{ij}(z) - g_{ij}(z)| < \varepsilon$. Then we can check that

$$|f - g| = \sup_{z \in S^1, |v| = 1} |(f(z) - g(z))(v)|$$

$$= \sup_{z \in S^1, |v| = 1} \left| \sum_{j=1}^n (f_{1j}(z) - g_{1j}(z))v_j \right| < \varepsilon \sqrt{n}.$$

So let $f$ be a clutching function. We know that $f(z)(v) \neq 0$ for every $z \in S^1$ and $|v| = 1$. Therefore, there is an $\varepsilon > 0$ such that $|f(z)(v)| > \varepsilon$ for all $z \in S^1$ and $|v| = 1$. Consider now the ball $B(f, \varepsilon/2)$ of center $f$ and radius $\varepsilon/2$ and let $g \in B(f, \varepsilon/2)$. So $|f(z)(v) - g(z)(v)| < \varepsilon/2$. But this implies that $|g(z)(v)| > \varepsilon/2$ for all $z \in S^1$ and $|v| = 1$. This means that $g$ is a clutching function and thus the set of clutching functions is open.

Now, let $f$ be a clutching function and let $\varepsilon > 0$ such that $B(f, \varepsilon)$ is contained in the clutching functions (we can choose such an $\varepsilon$ because the set is open). By
Theorem 7.2.13, for each \( f_{ij} \) there is a Laurent polynomial \( g_{ij} \) such that 

\[
|f_{ij}(z) - g_{ij}(z)| < \varepsilon/\sqrt{n}.
\]

Then \(|f - g| < \varepsilon\sqrt{n}/\sqrt{n} = \varepsilon\). So \( g \) is a clutching function and the map

\[
H(z, t) = tf(z) + (1 - t)g(z)
\]

is a homotopy via clutching functions from \( g \) to \( f \) (because \( B(f, \varepsilon) \) is convex). \( \square \)

7.2.15. So far, we have seen that any vector bundle in \( K^0(S^2) \) is equivalent to a linear combination of \( \tau_1 \) and \( H \), and hence it is in the image of \( \mu \).

Proof of Theorem 7.1.6. The morphism \( \mu \) is surjective by Proposition 7.2.14 and Lemma 7.2.11. To prove that \( \mu \) is injective we build a map

\[
\nu: K^0(S^2) \rightarrow \mathbb{Z}[H]/(H - 1)^2
\]

such that \( \mu \circ \nu = \text{id} \). The map \( \nu \) is constructed as follows: start with a vector bundle on \( K^0(S^2) \), take its clutching function, find a homotopic Laurent polynomial clutching function and reduce to a linear one. Thus the initial bundle is equivalent in \( K^0(S^2) \) to one of the form \([nH] + [m\tau_1]\) for some \( n, m \in \mathbb{Z} \). So we set \( \nu([H]) = H \) and \( \nu([\tau_1]) = 1 \). Thus map clearly satisfies that \( \mu \circ \nu = \text{id} \).

But to finish the proof we should check that \( \nu \) is well-defined. In other words, we have to see that if we have two equivalent vector bundles on \( K^0(S^2) \) and we do the previous ‘linearization’ procedure to reduce the corresponding clutching functions to linear ones, we get the same thing (somehow we need to check that it is independent of all choices). For instance, we will need to prove that homotopies between Laurent polynomial clutching functions can be replaced by homotopies that are a Laurent polynomial at each \( t \in [0, 1] \). We leave all the details as an exercise. \( \square \)

7.3. Some consequences of Bott periodicity

Corollary 7.3.1. \( \tilde{K}^0(S^2) \cong \mathbb{Z} \) generated by \((H - 1)\).

Proof. We have a split short exact sequence (see 5.1.2)

\[
0 \rightarrow \tilde{K}^0(S^2) \rightarrow K^0(S^2) \rightarrow K^0(\ast) = \mathbb{Z} \rightarrow 0.
\]

By Theorem 7.1.6, the term in the middle is \( \mathbb{Z}[H]/(H - 1)^2 \) and the third map sends \( aH + b \) to \( a + b \). So \( \tilde{K}^0(S^2) \), which is the kernel, is isomorphic to \( \mathbb{Z} \) and generated by \((H - 1)\). \( \square \)

Theorem 7.3.2 (Bott periodicity —standard form). For every \( X \) in \( \text{Top}_\ast \), the external product with \((H - 1)\) induces an isomorphism

\[
\tilde{K}^0(X) \xrightarrow{\cong} \tilde{K}^0(\Sigma^2 X) = \tilde{K}^{-2}(X).
\]

Proof. Recall from 6.2.2 that we have a commutative diagram

\[
\begin{array}{ccc}
K^0(X) \otimes K^0(S^2) & \cong & \tilde{K}^0(X) \otimes \tilde{K}^0(S^2) \oplus \tilde{K}^0(X) \oplus \tilde{K}^0(S^2) \oplus \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
K^0(X \times S^2) & \cong & \tilde{K}^0(X \wedge S^2) \oplus \tilde{K}^0(X) \oplus \tilde{K}^0(S^2) \oplus \mathbb{Z},
\end{array}
\]

Now, Theorem 7.1.5 states that the left map is an isomorphism, so the map in the middle is also an isomorphism. But, \( \tilde{K}^0(S^2) \cong \mathbb{Z} \) by Corollary 7.3.1. \( \square \)
Remark 7.3.3. As usual, we have an unreduced version of the previous statement by using $X_+$ for $X \in \text{Top}$. This gives an isomorphism $K^0(X) \cong K^{-2}(X)$ for every $X$ in $\text{Top}$.

Corollary 7.3.4. $\tilde{K}^0(S^{2n}) \cong \mathbb{Z}$ generated by $(H - 1)^n$ and $\tilde{K}^0(S^{2n+1}) = 0$.

Proof. It follows from Bott periodicity and the previous computations and is left as an exercise. $\square$

7.3.5. We finish with the list of all $K$-groups of the spheres (the computations are left as an exercise). By Bott periodicity it is enough to describe $K^0$ and $K^{-1}$. So, for every $n \geq 0$ we have

- $\tilde{K}^0(S^{2n}) \cong \mathbb{Z}$, $\tilde{K}^{-1}(S^{2n}) = 0$,
- $\tilde{K}^0(S^{2n+1}) = 0$, $\tilde{K}^{-1}(S^{2n+1}) \cong \mathbb{Z}$,
- $K^0(S^{2n}) \cong \mathbb{Z} \oplus \mathbb{Z}$, $K^{-1}(S^{2n}) = 0$,
- $K^0(S^{2n+1}) \cong \mathbb{Z}$, $K^{-1}(S^{2n+1}) \cong \mathbb{Z}$. 


LECTURE 8: THE COMPLEX $K'$-THEORY SPECTRUM

In this lecture we will show how to define the spectrum associated to the cohomology theory given by $\tilde{h}^\ast$. This allows to formulate yet another version of Bott periodicity. In the second part we will discuss the Hopf invariant one problem.

8.1. Reduced cohomology theories and spectra

8.1.1. Let $\tilde{h}^\ast$ be a reduced cohomology theory (see 6.1.2 for a precise definition). We will restrict to cohomology theories defined on pointed CW-complexes and we will also assume that they are additive, that is, they satisfy the wedge axiom:

$$\tilde{h}^n(\bigvee_{i \in I} X_i) \cong \prod_{i \in I} \tilde{h}^n(X_i).$$

8.1.2. For every $n \in \mathbb{Z}$ the functor $\tilde{h}^n$ satisfies the conditions of the Brown representability theorem (we need to be a bit careful here because Brown representability applies to functors from pointed connected CW-complexes, so we have to restrict to those spaces). So there is a (unique up to homotopy) pointed connected CW-complex $L_n$ and a natural equivalence

$$\tilde{h}^n(X) \cong [X, L_n],$$

for each pointed connected CW-complex $X$ (recall that $[-,-]$ denotes pointed homotopy classes of maps).

8.1.3. Let $E_n = \Omega L_{n+1}$, where $\Omega$ denotes the loop space functor, right adjoint to the suspension functor $\Sigma$. For any $X$ the suspension $\Sigma X$ is connected, so

$$\tilde{h}^{n+1}(\Sigma X) \cong [\Sigma X, L_{n+1}],$$

Since $\tilde{h}^\ast$ is a reduced cohomology theory $\tilde{h}^{n+1}(\Sigma X) \cong \tilde{h}^n(X)$, so

$$\tilde{h}^n(X) \cong \tilde{h}^{n+1}(\Sigma) \cong [\Sigma X, L_{n+1}] \cong [X, \Omega L_{n+1}] = [X, E_n],$$

where the third isomorphism is given by the adjunction between $\Sigma$ and $\Omega$. Thus, we can associate to $\tilde{h}^\ast$ the family of pointed CW-complexes $\{E_n\}_{n \in \mathbb{Z}}$ which satisfies that

$$[X, E_n] \cong \tilde{h}^n(X) \cong \tilde{h}^{n+1}(\Sigma X) \cong [\Sigma X, E_{n+1}] \cong [X, \Omega E_{n+1}],$$

for all pointed CW-complexes $X$. This implies that there is a homotopy equivalence

$$E_n \xrightarrow{\cong} \Omega E_{n+1}.$$

Definition 8.1.4. An $\Omega$-spectrum is a sequence of pointed CW-complexes $\{E_n\}_{n \in \mathbb{Z}}$ together with homotopy equivalences $\epsilon_n: E_n \to \Omega E_{n+1}$ for every $n \in \mathbb{Z}$.

So we have proved the following

Theorem 8.1.5. Every additive reduced cohomology theory $\tilde{h}^\ast$ on pointed CW-complexes determines an $\Omega$-spectrum $\{E_n\}_{n \in \mathbb{Z}}$ such that $\tilde{h}^n(X) = [X, E_n]$ for every $n \in \mathbb{Z}$. \hfill $\square$
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### 8.1.6. The converse is also true. Let $\{E_n\}_{n \in \mathbb{Z}}$ be an $\Omega$-spectrum and define $\overline{E}^n(X) = [X, E_n]_*$. Then $\overline{E}^*$ is a reduced cohomology theory. The homotopy invariance is easy to check and the suspension isomorphism is given by

$$\overline{E}^{n+1}(\Sigma) = [\Sigma X, E_{n+1}]_* \cong [X, \Omega E_{n+1}]_* \xrightarrow{\varepsilon_n^{-1}} [X, E_n]_* = \overline{E}^n(X).$$

This also implies that $\overline{E}^n(-)$ takes values in abelian groups, since $\overline{E}^n(X) \cong \overline{E}^{n+2}(\Sigma^2 X) = [\Sigma^2 X, E_{n+2}]_*$. Then $\overline{E}^n(X)$ and $\mathbb{Z}$ for every $n \geq 0$. The cohomology theory that it describes $\overline{E}^n(X) = [X, E_n]_*$ for every $n \in \mathbb{Z}$ form an additive reduced cohomology theory on pointed CW-complexes.

### 8.1.8. Let $G$ be an abelian group and let $K(G,n)$ be the associated Eilenberg–Mac Lane space. This space is characterized (up to homotopy) by the property that $\pi_k K(G,n) \cong G$ if $k = n$ and zero if $k \neq n$. There is a homotopy equivalence $K(G,n) \xrightarrow{\simeq} \Omega K(G, n+1)$. So the spaces $K(G,n)$ define an $\Omega$-spectrum $HG$ called the Eilenberg–Mac Lane spectrum associated to $G$. It is defined as $(HG)_n = K(G,n)$ for $n \geq 0$ and zero for $n < 0$. The cohomology theory that it describes $HG^*(X) = [X, K(G,n)]_* \cong \overline{H}^n(X; G)$ for $n \geq 0$, corresponds to singular cohomology with coefficients in $G$.

### 8.2. The spectrum $KU$

#### 8.2.1. Recall from the first lectures that the Grassmannian $G_k(\mathbb{C}^n)$ consists of all $k$-dimensional linear subspaces of $\mathbb{C}^n$. The canonical inclusion $\mathbb{C}^n \to \mathbb{C}^{n+1}$ that sends $(v_1, \ldots, v_n)$ to $(v_1, \ldots, v_n, 0)$ induces maps $i_n: G_k(\mathbb{C}^n) \to G_k(\mathbb{C}^{n+1})$.

We define $BU_k = \text{colim}_n \{G_k(\mathbb{C}^n), i_n\}$ as the colimit of the previous sequence. (Note that in the first lectures we were using the notation $G_k(\mathbb{C}^\infty)$ for $BU_k$.)

#### 8.2.2. Recall also that we proved the existence of a ‘universal’ vector bundle $E_k(\mathbb{C}^\infty)$ over $BU_k$ and that every vector bundle is a pullback of this one:

**Theorem 8.2.3.** Let $X \in \text{Top}$. There is a natural bijection $[X, BU_k] \cong \text{Vect}_k^b(X)$ that sends $f$ to the pullback $f^*(E_k(\mathbb{C}^n))$. $\square$
If we apply the previous theorem with \( k + 1 \) and \( X = BU_k \), we obtain a bijection \([BU_k, BU_{k+1}] \cong \text{Vect}_C^{k+1}(BU_k)\). So taking on the right-hand side the vector bundle \( E_k(\mathbb{C}^\infty) \oplus \tau_1 \) over \( BU_k \) gives a map

\[
i_k : BU_k \to BU_{k+1}
\]

such that \( i_k^*(E_{k+1}(\mathbb{C}^\infty) \cong E_k(\mathbb{C}^\infty) \oplus \tau_1 \). We define \( BU = \text{colim}_k \{BU_k, i_k\} \) as the colimit of the sequence given by the maps \( i_k \).

### Proposition 8.2.5

There is a split short exact sequence

\[
0 \to \tilde{K}(X) \to K^0(X) \to [X, \mathbb{Z}] \to 0.
\]

In particular \( K^0(X) \cong \tilde{K}(X) \oplus [X, \mathbb{Z}] \).

**Proof.** To prove the result it is enough to find a section to the map on the right. Let \( f : X \to \mathbb{N} \). Since \( X \) is compact \( f(X) \) is compact in \( \mathbb{N} \) and hence finite. So suppose that \( f(X) = \{n_1, \ldots, n_r\} \). Then \( X = X_1 \coprod \cdots \coprod X_r \), where each \( X_i = f^{-1}(n_i) \). We define a bundle over \( X \) by taking trivial bundles \( \tau_n \) at each \( X_i \). This defines a map \( \varphi : [X, \mathbb{N}] \to \text{Vect}_C(X) \) that satisfies \( d \circ \varphi = \text{id} \).

Now, using the universal property of the group completion there exists a map \( \overline{\varphi} : [X, \mathbb{Z}] \to K^0(X) \) that satisfies \( \overline{d} \circ \overline{\varphi} = \text{id} \). The map \( \overline{\varphi} \) is the required section. \( \square \)

### Corollary 8.2.6

If \( X \in \text{Top}_* \) is connected, then \( \tilde{K}(X) \cong \tilde{K}^0(X) \).

**Proof.** Consider the following commutative diagram of split short exact sequences

\[
\begin{array}{ccc}
0 & \to & \tilde{K}(X) \\
\downarrow & & \downarrow \text{id} \\
0 & \to & K^0(X)
\end{array}
\begin{array}{cccc}
\to & [X, \mathbb{Z}] & \to & 0 \\
\text{id} & \downarrow & & \downarrow \overline{i}^* \\
\to & [*, \mathbb{Z}] & \to & 0
\end{array}
\]

where \( i : * \to X \) is the inclusion of the basepoint. If \( X \) is connected, then \( i^* \) is an isomorphism and hence \( \tilde{K}(X) \cong \tilde{K}^0(X) \). \( \square \)

### 8.2.7

Consider the sets \( \text{Vect}_C^k(X) \) and define for every \( k \geq 0 \) a function

\[
t_k : \text{Vect}_C^k(X) \to \text{Vect}_C^{k+1}(X)
\]
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Proposition 8.2.8. For every $X$ in Top we have that $\text{Vect}_s(X) \cong \hat{K}(X)$.

Proof. For each $k \geq 0$ define a map $\varphi_k : \text{Vect}_c^k(X) \to \hat{K}(X)$ as $\varphi_k([E]) = [E] - [\tau_k] \in \hat{K}(X)$. Then $\varphi_{k+1} t_k([E]) = \varphi_k([E])$ for every $k$, so by the universal property of the colimit, there is a map $\varphi : \text{Vect}_s(X) \to \hat{K}(X)$ and a commutative triangle

\[
\begin{array}{ccc}
\text{Vect}_c^k(X) & \longrightarrow & \text{Vect}_s(X) \\
\varphi_k & & \varphi \\
& \hat{K}(X) & \\
\end{array}
\]

Now using the fact that for every bundle $E$ we can find another bundle $E'$ such that $E \oplus E' \cong \tau_n$ for some $n$, one checks that $\varphi$ is injective and surjective. \hfill \Box

Proposition 8.2.9. For every $X$ in Top there is an isomorphism $\hat{K}(X) \cong [X, BU]$.

Proof. By Theorem 8.2.3, we know that $\text{Vect}_c^k(X) \cong [X, BU_k]$. The previously defined maps $t_k : \text{Vect}_c^k(X) \to \text{Vect}_c^{k+1}(X)$ and $i_k : BU_k \to BU_{k+1}$ are compatible with this isomorphism. So we have an isomorphism after taking colimits

$$\text{colim}_k \text{Vect}_c^k(X) \cong \text{colim}_k [X, BU_k]$$

The left-hand side is $\text{Vect}_s(X)$, which by Proposition 8.2.8 is isomorphic to $\hat{K}(X)$. The fact that $X$ is compact and that the maps $i_k$ are embeddings (see exercise sheet 8) the right-hand side is isomorphic to $[X, \text{colim}_k BU_k] = [X, BU]$.

Corollary 8.2.10. If $X \in \text{Top}$, then $K^0(X) \cong [X, BU \times Z]$. If $X \in \text{Top}_*$ and $X$ is connected, then $\hat{K}^0(X) \cong [X, BU]$.

Proof. By Proposition 8.2.5 we have a splitting $K^0(X) \cong \hat{K}(X) \oplus [X, Z]$. This fact, together with Proposition 8.2.9 implies that $[X, BU \times Z]$. The second part follows from Corollary 8.2.6 and Proposition 8.2.9. \hfill \Box

Corollary 8.2.11. Let $X \in \text{Top}_*$ such that the inclusion $i : * \to X$ is a cofibration (e.g., if $X$ is a CW-complex). Then $\hat{K}^0(X) \cong [X, BU \times Z]_*$. 

Proof. We need to show that $[X, BU \times Z]_*$ is the kernel of the map

$$K^0(X) \cong [X, BU \times Z] \overset{i^*}{\longrightarrow} [*, BU \times Z] \cong K^0(*)$$

Let $j : [X, BU \times Z]_* \to [X, BU \times Z]$ be the natural inclusion. If $f \in [X, BU \times Z]_*$, then $i^* j(f)$ is zero in $[* , BU \times Z]$. So $[X, BU \times Z]_* \subseteq \ker i^*$.

To prove the converse, let $g \in [X, BU \times Z]$ and suppose that $i^* j(g)$ is zero. Since $BU$ is connected, there is a homotopy between the basepoint of $BU$ and $g_1(x_0)$, where $x_0$ denotes the basepoint of $X$. So we can build a homotopy

$$\alpha : \{x_0\} \times I \longrightarrow BU \times Z$$
between \((g_1(x_0), 0)\) and \((*, 0)\), where here \(*\) is the basepoint of \(BU\). Now consider the following diagram

\[
X \times \{0\} \cup \{x_0\} \times I \xrightarrow{(g, \alpha)} BU \times \mathbb{Z} \\
\downarrow H \\
X \times I.
\]

Since \(* \to X\) is a cofibration, there is a lifting \(H\), giving a homotopy between \(g(x) = H(x, 0)\) and \(H(x, 1)\) which is a pointed map, since \(H(x_0, 1) = \alpha(x_0, 1) = (*, 0)\). □

8.2.12. The family of spaces \(E_{2n} = BU \times \mathbb{Z}\) and \(KU_{2n+1} = \Omega BU\) for \(n \in \mathbb{Z}\) have the property that \(KU_{2n-1} = \Omega BU = \Omega (BU \times \mathbb{Z}) = \Omega KU_{2n}\).

By the standard form of Bott periodicity (see Theorem 7.3.2) we know that \(\widetilde{\mathcal{K}}^0(X) \cong \mathcal{K}^0(\Sigma^2 X)\), hence Corollary 8.2.11 shows that \(\widetilde{\mathcal{K}}^0(\Sigma^2 X) \cong \mathcal{K}^0(X)\).

8.2.13. Therefore the sequence \(\{KU_n\}_{n \in \mathbb{Z}}\) defines an \(\Omega\)-spectrum called the complex \(K\)-theory spectrum, and hence a reduced cohomology theory, by Theorem 8.1.7. If \(X\) is a pointed finite CW-complex, then \(\widetilde{KU}(X) \cong \mathcal{K}(X)\).

8.2.14. The existence of a homotopy equivalence \(BU \times \mathbb{Z} \simeq \Omega^2 BU\) is equivalent to Bott periodicity. In fact, this equivalence will be proved in the last part of the course, by using simplicial methods.

**Theorem 8.2.15** (Bott periodicity — topological version). *There is a homotopy equivalence \(BU \times \mathbb{Z} \simeq \Omega^2 BU\).*

8.3. The Hopf invariant one problem

8.3.1. A *multiplication* for the sphere \(S^n\) is a continuous map \(\mu: S^n \times S^n \to S^n\) with a two-sided unit \(e \in S^n\) such that \(\mu(x, e) = \mu(e, x) = x\) for all \(x \in S^n\). For the values \(n = 0, 1, 3\) and 7 such a multiplication exists, and it is given by the multiplication in \(\mathbb{R}, \mathbb{C}, \mathbb{H}\) and \(\mathcal{O}\), respectively. We would like to know if these are the only possible cases. Indeed, this is true.

**Theorem 8.3.2.** *\(S^n\) admits a multiplication if and only if \(n = 0, 1, 3\) or 7.*

We will not prove this theorem, but we will show how it is implied by the existence of Hopf invariant one maps between certain spheres. This result is important because of the following

**Proposition 8.3.3.** If \(\mathbb{R}^n\) is a division algebra, then \(S^{n-1}\) admits a multiplication.

**Proof.** We can assume that the multiplication on \(\mathbb{R}^n\) has an identity that is a unit vector (if this is not the case one can always modify the multiplication to obtain one with this property). The multiplication on \(S^{n-1}\) is defined by sending \((x, y)\) to \(xy\) divided by its norm. □

Hence, from Theorem 8.3.2, we deduce that \(\mathbb{R}^n\) is a division algebra if and only if \(n = 1, 2, 4\) or 8.
8.3.4. We can use the computations that we have made of the \( K \)-groups of the spheres to show that multiplications do not exist for spheres of even dimension. We will use that \( K^0(S^{2n}) = \mathbb{Z} \oplus \mathbb{Z} \) (generated by the trivial bundle and \( (H - 1)^n \)), but we will also need the ring structure of \( K^0(S^{2n}) \), which can be deduced from the following proposition. Recall that for \( X \in \text{Top}_* \) we have defined the external product \( \tilde{K}^0(X) \otimes \tilde{K}^0(X) \to \tilde{K}^0(X \wedge X) \). We can compose this with a map induced by the diagonal \( \Delta: X \to X \times X \) to get a product map 

\[
\tilde{K}^0(X) \otimes \tilde{K}^0(X) \to \tilde{K}^0(X \wedge X) \to \tilde{K}^0(X)
\]

**Proposition 8.3.5.** Let \( X \in \text{Top}_* \) and let \( X = A \cup B \), where \( A \) and \( B \) are closed contractible subspaces of \( X \). Then the product map \( \tilde{K}^0(X) \otimes \tilde{K}^0(X) \to \tilde{K}^0(X) \) is trivial.

**Proof.** Since \( A \) and \( B \) are contractible \( \tilde{K}^0(X) \cong \tilde{K}^0(X/A) \) and \( \tilde{K}^0(X) \cong \tilde{K}^0(X/B) \) by Corollary 6.1.6. The external product defines a map 

\[
\tilde{K}^0(X/A) \otimes \tilde{K}^0(X/B) \to \tilde{K}^0(X/A \wedge X/B)
\]

and one can check that \( X/A \wedge X/B \cong X \times X/(A \times X) \cup (X \times B) = W \) and that the diagonal induces a map \( X/A \cup B \to W \). So, we have a commutative diagram

\[
\begin{array}{ccc}
\tilde{K}^0(X/A) \otimes \tilde{K}^0(X/B) & \longrightarrow & \tilde{K}^0(W) \\
\downarrow & & \downarrow \\
\tilde{K}^0(X) \otimes \tilde{K}^0(X) & \longrightarrow & \tilde{K}^0(X \wedge X) \longrightarrow \tilde{K}^0(X)
\end{array}
\]

where the right up corner is zero (because \( A \cup B = X \)). Since the left map is an isomorphism the bottom composition (which is the product) is zero. \( \square \)

**Corollary 8.3.6.** \( K^0(S^{2n}) \cong \mathbb{Z}[\gamma]/(\gamma^2) \).

**Proof.** Let \( \gamma = (H - 1)^n \). Then \( \gamma \in \tilde{K}^0(S^{2n}) \) and we know, by Proposition 8.3.5, that \( \gamma^2 = 0 \) (by taking \( A \) the closed upper hemisphere and \( B \) the closed lower hemisphere of \( S^{2n} \)). \( \square \)

**Proposition 8.3.7.** The sphere \( S^{2n} \) does not admit a multiplication for \( n \geq 1 \).

**Proof.** Recall that Bott periodicity for reduced \( K \)-theory (see Theorem 7.3.2) states that we have an isomorphism 

\[
\tilde{K}^0(X) \otimes \tilde{K}^0(S^2) \cong \tilde{K}^0(X \wedge S^2).
\]

We can iterate this isomorphism several times, by replacing \( X \) by \( X \wedge S^2 \), and we get an isomorphism 

\[
\tilde{K}^0(X) \otimes \tilde{K}^0(S^{2n}) \cong \tilde{K}^0(X \wedge S^{2n}).
\]

for every \( n \geq 1 \), and a corresponding isomorphism for the unreduced case 

\[
K^0(X) \otimes K^0(S^{2n}) \cong K^0(X \times S^{2n}).
\]

Suppose now that \( \mu: S^{2n} \times S^{2n} \to S^{2n} \) is a multiplication. Applying \( K^0 \) gives a map 

\[
\mathbb{Z}[\gamma]/(\gamma^2) \cong K^0(S^{2n}) \xrightarrow{\mu^*} K^0(S^{2n} \times S^{2n}) \cong K^0(S^{2n}) \otimes K^0(S^{2n}) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2).
\]
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Since $\mu$ has a two-sided unit (given by map $e: * \to S^{2n}$) we have a commutative diagram

$$
\begin{array}{ccc}
S^{2n} \times \{\ast\} & \xrightarrow{1 \times e} & S^{2n} \times S^{2n} \\
\downarrow \mu & & \downarrow \leftarrow \downarrow 1 \times e \ \\
S^{2n} & & * \times S^{2n}
\end{array}
$$

which induces a commutative diagram

$$
\begin{array}{ccc}
K^0(S^{2n}) & \xleftarrow{\mu^*} & K^0(S^{2n} \times S^{2n}) \\
\downarrow & & \downarrow \end{array}
\quad \begin{array}{ccc}
K^0(S^{2n}) \hookrightarrow K^0(S^{2n}) \\
\downarrow & & \downarrow
\end{array}
\quad \begin{array}{ccc}
K^0(S^{2n}) & \xrightarrow{\mu^*} & K^0(S^{2n})
\end{array}
$$

where the map to the left sends $\alpha$ to $\gamma$ and $\beta$ to 0, and the map to the right sends $\alpha$ to 0 and $\beta$ to $\gamma$. This forces $\mu^*(\gamma)$ to be of the form $\alpha + \beta + t\alpha\beta$. Hence

$$0 = \mu^*(\gamma^2) = \alpha^2 + \beta^2 + t^2\alpha^2\beta^2 + 2t\alpha\beta + 2t\alpha^2\beta + 2\alpha\beta = 2\alpha\beta.$$ But $2\alpha\beta$ cannot be zero, so $\mu$ cannot be a multiplication.

8.3.8. Things became more involved for spheres of odd dimension. Here is where the Hopf invariant appears. Suppose that we have a map $g: S^{n-1} \to S^{n-1} \to S^{n-1}$ for $n$ an even number. We can decompose the sphere $S^{2n-1}$ as follows

$$S^{2n-1} = \partial(D^{2n}) = \partial(D^n \times D^n) = S^{n-1} \times D^n \cup D^n \times S^{n-1},$$

where $D^n$ denotes the $n$-dimensional disk in $\mathbb{R}^n$. We can define maps

$$
\begin{array}{ccc}
S^{n-1} \times D^n & \to & D^n \\
(x, y) & \mapsto & |y|g(x, y/|y|)
\end{array}
\quad \begin{array}{ccc}
D^n \times S^{n-1} & \to & D^n \\
(x, y) & \mapsto & |x|g(x/|x|, y).
\end{array}
$$

They coincide in the intersection of the domains so we give a map

$$\hat{g}: S^{2n-1} = S^{n-1} \times D^n \cup D^n \times S^{n-1} \to D^n \cup D^n = S^n.$$ The map $\hat{g}$ is called the Hopf construction of the map $g$.

8.3.9. We are interested in odd spheres, so let now $g: S^{2n-1} \times S^{2n-1} \to S^{2n-1}$, build the corresponding $\hat{g}: S^{4n-1} \to S^{2n}$ and take the mapping cone

$$S^{4n-1} \xrightarrow{\hat{g}} S^{2n} \to C(\hat{g})$$

obtained by taking the cone on $S^{4n-1}$, which is $D^{4n}$, and gluing it to $S^{2n}$ via $\hat{g}$. Then, applying $K^0$ gives a split short exact sequence

$$0 \to \tilde{K}^0(S^{4n}) \to \tilde{K}^0(C(\hat{g})) \to \tilde{K}^0(S^{2n}) \to 0$$

since $\tilde{K}^0$ of odd degree spheres is zero. So the group in the middle is $\mathbb{Z} \oplus \mathbb{Z}$ generated by $\alpha$ and $\beta$, where $\alpha$ is the image of the generator of $\tilde{K}^0(S^{4n})$ (hence $\alpha^2 = 0$) and $\beta$ is some class mapped to the generator of $\tilde{K}^0(S^{2n})$ (hence $\beta^2$ is mapped to 0). By exactness, $\beta^2$ is in the image of the second map, so there is an integer $H(\hat{g})$ such that $\beta^2 = H(\hat{g})\alpha$. The integer $H(\hat{g})$ is called the Hopf invariant.

One has to be a bit careful here, because we have to check that $\beta$ is well-defined. We could have replaced $\beta$ by $\beta + ta$. Then $\beta^2$ would be $\beta^2 + 2t\alpha\beta$. We claim that $\alpha\beta = 0$. Indeed, $\alpha\beta$ is mapped to 0 in $\tilde{K}^0(S^{2n})$, so $\alpha\beta = ka$ for some $k$. But $\alpha\beta^2 = k\alpha\beta = k^2\alpha$. On the other hand, $\alpha\beta^2 = H(\hat{g})\alpha^2 = 0$ so $k^2\alpha = 0$ and hence $k = 0$. So the Hopf invariant is independent of $\beta$. 


Note, however, that the Hopf invariant is not independent of the choice of α since we could choose −α and this would change \(H(\bar{g})\) by \(-H(\bar{g})\). So, by convention, we will always assume that the Hopf invariant is non-negative, and hence independent of \(\alpha\) and \(\beta\).

**Proposition 8.3.10.** If \(g: S^{2n-1} \times S^{2n-1} \to S^{2n-1}\) is a multiplication, then \(\bar{g}: S^{4n-1} \to S^{2n}\) has Hopf invariant one.

**Proof.** As we have seen, the map \(\Phi: (D^{2n} \times D^{2n}, \partial (D^{2n} \times D^{2n})) \to (C(\bar{g}), S^{2n})\).

Recall that \(\bar{g}\), and therefore also \(\Phi\), send \(S^{2n-1} \times D^{2n}\) into \(D^{2n}_+\) and \(D^{2n} \times S^{2n-1}\) into \(D^{2n}\). Consider the following diagram

\[
\begin{array}{ccc}
\tilde{K}^0(C(\bar{g})) & \to & \tilde{K}^0(C(\tilde{g})) \\
\Phi^\ast \otimes \Phi^\ast & & \Phi^\ast \\
\tilde{K}^0(C(\bar{g})/D^{2n}_+) & \to & \tilde{K}^0(C(\bar{g})/D^{2n}) \\
\Phi^* & \cong & \Phi^* \\
\tilde{K}^0(D^{2n}_+ \times D^{2n}_+ / S^{2n-1} \times D^{2n}) & \to & \tilde{K}^0(D^{2n}_+ \times D^{2n} / \partial (D^{2n}_+ \times D^{2n})) \\
\cong & & \cong \\
\tilde{K}^0(D^{2n}_+ \times S^{2n-1} / S^{2n-1} \times S^{2n-1}),
\end{array}
\]

where the diagonal map is an isomorphism by Bott periodicity. Take now the composite

\[\tilde{K}^0(C(\bar{g})/D^{2n}_+) \xrightarrow{\Phi^*} \tilde{K}^0(D^{2n} \times D^{2n} / S^{2n-1} \times D^{2n}) \to \tilde{K}^0(D^{2n} \times S^{2n-1} \times \ast).
\]

Due to how we have defined the map \(\bar{g}\), the previous composite is induce by the map of pairs

\[(D^{2n} \times \ast, S^{2n-1} \times \ast) \cong (D^{2n}, S^{2n-1}) \to (S^{2n}, D^{2n}_+) \to (C(\bar{g}), D^{2n}_+),
\]

where the last two maps on the right are inclusions. But then \(\beta\) in \(\tilde{K}^0(C(\bar{g}))\) maps to a generator of \(\tilde{K}^0(S^{2n}/D^{2n}_+) \cong \tilde{K}^0(D^{2n}_+, S^{2n-1})\). Similarly, one can show that \(\beta \otimes \beta\) maps to a generator of the bottom left-hand group. Thus, \(\beta \otimes \beta\) maps to a generator of the bottom right-hand group and then the Hopf invariant has to be, by our convention, equal to 1.

**8.3.11.** By Proposition 8.3.10, to prove Theorem 8.3.2 it is enough to show that there is no map \(S^{4n-1} \to S^{2n}\) of Hopf invariant 1 unless \(n = 1, 2\) or 4. This is a non-trivial statement and uses Adams operations on \(K\)-theory and the splitting principle.