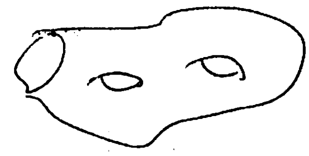
Utrecht, L2

Σ = orientable 2-surface, possibly with boundary

G = connected Lie group $\qquad \Sigma =$ 

$\mathfrak{g} = \mathrm{Lie}(G)$

$\mathfrak{g}$ is <u>quadratic</u>

$(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ nondegenerate invariant symmetric bilinear form

Ex: • $\mathfrak{g}$ = semisimple $\quad (x,y) = \mathrm{Tr}\, \mathrm{ad}_x \mathrm{ad}_y$ Killing form

• $(\mathfrak{g}, \mathfrak{a}, \mathfrak{a}^*)$ a <u>Manin triple</u>

$\mathfrak{a}, \mathfrak{a}^*$ = isotropic Lie subalgebras $\quad \mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^*$

Sub ex: • $\mathfrak{a}^*$ = abelian, $\quad x \in \mathfrak{a}, \xi \in \mathfrak{a}^* \quad [x,\xi] = \mathrm{ad}_x^* \xi$

<u>Remark</u>: $(\cdot,\cdot) \Rightarrow \quad \mathfrak{g} \cong \mathfrak{g}^*$ can think of coadjoint orbits $\subset \mathfrak{g}$

↳ a canonical element $\eta \in (\Lambda^3 \mathfrak{g}^*)^G \cong (\Omega^3(G))^{G\times G}$

Cartan 3-form

$$\eta(x,y,z) = (x,[y,z]) \qquad x,y,z \in \mathfrak{g}$$

$$\eta = \frac{1}{6}(\theta^L,[\theta^L,\theta^L]) = \frac{1}{6}(\theta^R,[\theta^R,\theta^R])$$

$P \to \Sigma$ trivial principal G-bundle $\quad P = G \times \Sigma$

$\mathcal{A}(\Sigma,\mathfrak{g}) = \Omega^1(\Sigma,\mathfrak{g})$ the space of connections on Σ

The group $\Sigma G = \mathrm{Map}(\Sigma, G)$ acts on $\mathcal{A}(\Sigma,\mathfrak{g})$ by gauge transformations

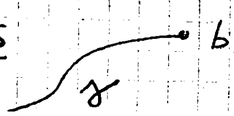
$$g : A \longmapsto \mathrm{Ad}_{g^{-1}} A + g^*\theta^L = g^{-1}Ag + g^{-1}dg$$

Lie algebra action $\Sigma\mathfrak{g}$:

$$x : A \longmapsto -[x,A] + dx$$

<u>Curvature</u> $\quad F_A = dA + \frac{1}{2}[A,A]$

$$g: F_A \mapsto g^{-1}F_A g, \qquad x: F_A \longmapsto -[x,F_A]$$

<u>Holonomies</u> 

$$\mathrm{Hol}(A^g,\gamma) = g(a)^{-1}\,\mathrm{Hol}(A,\gamma)\, g(b)$$

Atiyah-Bott symplectic form on $\mathcal{A}$

$$\omega = \frac{1}{2}\int_\Sigma (\delta A ; \delta A) \qquad \delta = \text{de Rham differential on } \mathcal{A}$$

Prop The action of ΣG is Hamiltonian

Proof
$$\iota(x_{\mathcal{A}})\omega = \int_\Sigma (-[x,A]+dx, \delta A) =$$
$$= \delta\left(\int_{\partial\Sigma}(x,A) + \int_\Sigma (x, dA + \tfrac{1}{2}[A,A])\right)$$

1. $\partial\Sigma = \emptyset$ closed surface

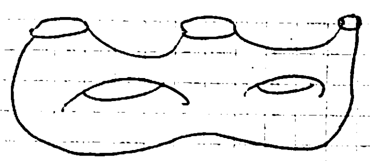
$\mu : \mathcal{A}(\Sigma, G) \longrightarrow \Omega^2(\Sigma, \mathfrak{g})$, $A \mapsto F_A$

reduced space

$$\mathcal{M}(\Sigma, G) = \mu^{-1}(0)/\Sigma G = \{A \in \Omega^1(\Sigma,\mathfrak{g}) ; F_A = 0\}/A \sim A^g$$

symplectic

2. $\partial\Sigma = \underbrace{S^1 \cup \ldots \cup S^1}_{k \text{ times}}$



reduction for $\Sigma_\circ G = \{g \in \Sigma G ; g(p) = e, p \in \partial\Sigma\}$

$$\mathcal{N}(\Sigma, G) = \{A \in \Omega^1(\Sigma, \mathfrak{g}) ; F_A = 0\}/A \sim A^g \text{ for } g \in \Sigma_\circ G$$

symplectic ∇ assume G is simply connected

$\circlearrowleft$

$$LG^k \cong \Sigma G / \Sigma_\circ G$$

3. ~~$\mathcal{N}/\Sigma$~~ $\mathcal{N}(\Sigma, G)/LG^k$ is a Poisson space

G = compact, connected and simply connected

$\Rightarrow \Omega^1(S^1, \mathfrak{g})/LG \cong \Delta_W$ the Weyl alcove

$A \mapsto g^{-1}Ag + g^{-1}dg$ Ex: $G = SU(2)$

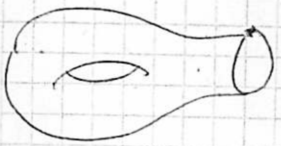
$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \qquad \theta \in [0, \pi]$$

for $k = 1$ $\quad \pi : \mathcal{N}(\Sigma, G)/LG \longrightarrow \Delta_W$

is a Poisson map with PB on $\Delta_W = 0$

fibers = symplectic leaves

Ex:



$$\{[A] \in \mathcal{N}(\Sigma, G);\ \cancel{A|_{\partial\Sigma} = 0}\}/_{LG} \cong \mathcal{M}(\Sigma, G)$$

$$\underline{\pi(A) = 0}$$

$\underline{4}$ $\quad \partial\Sigma \cong S^1$ choose a base point $p \in S^1$

$$\mathcal{L}(\Sigma, G) = \{A \in \Omega^1(\Sigma, \mathfrak{g});\ F_A = 0\}/_{A \sim A^g},\ \overbrace{g \in \Sigma G,\ g(p) = e}^{\Sigma_p G}$$

$\circlearrowleft$

$G \cong \Sigma G / \Sigma_p G$ $\quad$ not ~~..~~ symplectic, not Poisson

<u>but</u> $\Sigma_p G$ <u>acts freely</u> $\Rightarrow \mathcal{L}(\Sigma, G)$ smooth and <u>finite-dimensional</u>

$$Hol(A^g;\ p \xrightarrow{\gamma} p') = \underset{\substack{\parallel \\ e}}{g(p)}^{-1}\, Hol(A, \sim)\, g(p')$$

$$\Rightarrow g(p') = e$$