LECTURES ON MULTIPLICATIVE STRUCTURES; OUTLINE

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1. Introduction

We will study geometric structures coupled with group-like structures (Lie groupoids), and their role in Poisson geometry; focu on the interplay with infinitesimal geometry and their symmetries.

Multiplicative structures will lead us to several objets of interest in Poisson geometry, including Lie (bi)algebroids, Courant brackets/Courant algebroids, Dirac structures, moment maps...

We will somewhat follow the historical evolution: Poisson-Lie groups, symplectic groupoids, Poisson groupoids (contact, Jacobi groupoids); Multiplicative multivector fields, multiplicative forms (relation with Dirac structures); Motivation and applications include integrable systems, quantization, symmetries and moment maps (including G-valued), equivariant cohomology, Cartan's work... So there will be many connections with other minicourses.

The plan is to proceed as follows.

2. First examples of multiplicative structures

2.1. **Poisson-Lie groups.** Definition of multiplicative Poisson structures on Lie groups G (Poisson Lie groups): the multiplication map $m : G \times G \to G$ is a Poisson map.

Basic properties, first examples (r-matrices).

2.2. **Lie groupoids.** We will need to consider more than just groups. Lie groupoids naturally appear in symplectic/Poisson geometry, as we will see.

A *Lie groupoid* is a pair of smooth manifold $(\mathcal{G}, M)$ equipped with the following structure maps:

1. surjective submersions $\mathsf{s}, \mathsf{t} : \mathcal{G} \to M$, called *source* and *target maps*;
2. a smooth *multiplication* map $m : \mathcal{G}_{(2)} \to \mathcal{G}$, defined on the space $\mathcal{G}_{(2)} = \{(g,h) \,|\, \mathsf{s}(g) = \mathsf{t}(h)\}$ of *composable pairs*, also denoted by $m(g,h) = g.h$;
3. a diffeomorphism $i : \mathcal{G} \to \mathcal{G}$, called *inversion*; we often write $i(g) = g^{-1}$;
4. an embedding $\varepsilon : M \to \mathcal{G}$, called *unit map*; we often write $\epsilon(x) = 1_x$, or simply x.

The structure maps must satisfy the following properties:

- composition law: $\mathsf{s}(gh) = \mathsf{s}(h)$, $\mathsf{t}(gh) = \mathsf{t}(g)$;
- associativity law: $(gh)k = g(hk)$
- law of units: $\mathsf{s}(\varepsilon(x)) = \mathsf{t}(\varepsilon(x)) = x$, and $g\varepsilon(\mathsf{s}(g)) = \varepsilon(\mathsf{t}(g))g = g$
- law of inverses: $\mathsf{s}(g^{-1}) = \mathsf{t}(g)$, $\mathsf{t}(g^{-1}) = \mathsf{s}(g)$ and $g^{-1}g = \varepsilon(\mathsf{s}(g))$, $gg^{-1} = \varepsilon(\mathsf{t}(g))$.

We will use the notation $\mathcal{G} \rightrightarrows M$, and sometimes refer to M as the *space of objects* and $\mathcal{G}$ as the *space of arrows* [groupoids are particular cases of categories].

Example 2.1.

- *Manifolds $M \rightrightarrows M$, where source and target maps are the identity.*
- *Pair and fundamental groupoids: $M \times M \rightrightarrows M$, $\Pi(M) \rightrightarrows M$.*
- *Lie groups $G \rightrightarrows \{*\}$ are Lie groupoids over a point.*
- *Action groupoids: consider an action $G \curvearrowright M$. The product $\mathcal{G} = G \times M$ has a groupoid structure over M given by*
$$\mathsf{s}(g,x) = x, \;\; \mathsf{t}(g,x) = g.x, \;\; m((g,x),(h,y)) = (gh,y).$$
We denote action groupoids by $G \ltimes M$. In particular, $T^*G = G \times \mathfrak{g}$ has a grupoid structure coming from the adjoint action.
- *Tangent Lie groupoids: For a Lie groupoid $\mathcal{G} \rightrightarrows M$, we have a Lie groupoid $T\mathcal{G} \rightrightarrows TM$, with source (resp. target) map given by $T\mathsf{s} : T\mathcal{G} \longrightarrow TM$ (resp. $T\mathsf{t} : T\mathcal{G} \longrightarrow TM$), and multiplication defined by $Tm : T\mathcal{G}^{(2)} = (T\mathcal{G})^{(2)} \longrightarrow T\mathcal{G}$. Cotangent bundles of Lie groupoids also carry natural Lie groupoid structures, we will see this later.*

There are two main ingredients of a Lie groupoid: its *orbits*
$$\mathcal{O}_x = \{\mathsf{t}(g) \,|\, \forall g, \; \mathsf{s}(g) = x\},$$
and *isotropy groups*
$$\mathcal{G}_x = \{g \,|\, \mathsf{s}(g) = \mathsf{t}(g) = x\}.$$
The orbits decompose the base M into "leaves".

On a Lie groupoid $\mathcal{G} \rightrightarrows M$, one may still define the operations of left and right multiplications: if $\mathsf{s}(g) = x$ and $\mathsf{t}(g) = y$, then
$$l_g : \mathsf{t}^{-1}(x) \to \mathsf{t}^{-1}(y), \qquad r_g : \mathsf{s}^{-1}(y) \to \mathsf{s}^{-1}(x).$$

A Lie groupoid $\mathcal{G} \rightrightarrows M$ also encodes a group of "symmetries" of the base manifold: a *bisection* of $\mathcal{G}$ is a submanifold $N \subset M$ if the restrictions $\mathsf{t}|_N$ and $\mathsf{s}|_N$ are diffeomorphisms from N to M; the space of bisections has a natural group operation, and give rise to diffeomorphisms of M.

Lie groupoid morphisms and subgroupoids are defined in a natural way.

The simplest multiplicative objects on a Lie groupoid $\mathcal{G} \rightrightarrows M$ are *multiplicative functions*: there are functions $f : \mathcal{G} \to \mathbb{R}$ satisfying
$$f(gh) = f(g) + f(h),$$
i.e., groupoid morphisms into $\mathbb{R} \rightrightarrows \{*\}$.

2.3. **Symplectic groupoids.** The cotangent bundle of a Lie group is the prototypical example of a symplectic groupoid: it is naturally a symplectic manifold and can be seen as an action groupoid $G \ltimes \mathfrak{g}^*$ (adjoint action). We will see how symplectic and groupoid structures are compatible.

A Lie groupoid $\mathcal{G} \rightrightarrows M$ equipped with a symplectic form $\omega \in \Omega^2(\mathcal{G})$ is called a *symplectic groupoid* if the graph
$$\Gamma_m = \{(g,h,gh) \,|\, (g,h) \in \mathcal{G}_{(2)}\}$$
of the groupoid multiplication is a lagrangian submanifold of
$$(\mathcal{G},\omega) \times (\mathcal{G},\omega) \times (\mathcal{G},-\omega);$$

We will commonly use the notation $\overline{\mathcal{G}}$ for $\mathcal{G}$ equipped with $-\omega$.

The definition comes from quantization...

Several properties follow:

(1) the embedding $\varepsilon : M \to \mathcal{G}$ is lagrangian;
(2) the inversion $i : \mathcal{G} \to \mathcal{G}$ is anti-symplectomorphism;
(3) for any $g \in \mathcal{G}$, $\ker(T\mathsf{s})_g = \ker(T\mathsf{t})_g^\omega$; in particular $\{\mathsf{s}^* C^\infty(M), \mathsf{t}^* C^\infty(M)\} = 0$;
(4) There is a unique Poisson structure on M for which $\mathsf{t} : \mathcal{G} \to M$ is a Poisson map [in this case, s is an anti-Poisson map...

This last result reveals a key property of symplectic groupoids: they are *symplectic realizations* of their bases.

Some simple examples are:

Example 2.2. *For M symplectic, $M \times \overline{M}$, or the fundamental groupoid $\Pi(M)$, are symplectic groupoids; the induced Poisson structure on M is the same symplectic structure.*

A cotangent bundle $T^*M \rightrightarrows M$, with groupoid structure given by fibrewise addition, is a symplectic groupoid; the induced Poisson structure on M is zero.

The key example is $T^*G \rightrightarrows \mathfrak{g}^*$, with groupoid structure given by action $T^*G = G \ltimes \mathfrak{g}^*$; the induced Poisson structure on $\mathfrak{g}^*$ is the linear/Lie (KKS)-Poisson one.

Given Poisson manifold (M, π), can one find a symplectic groupoid $\mathcal{G} \rightrightarrows M$ so that t is a Poisson map? Is it unique? This is a type of integration problem (for $\mathfrak{g}^*$, it leads to a proof of Lie's third theorem).

Applications: Hamiltonian actions and moment maps; quantization...

2.4. **Poisson groupoids.** One can consider Poisson structures compatible with Lie groupoids and unify Poisson Lie groups and symplctic groupoids.

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid equipped with a Poisson structure $\pi \in \Gamma(\wedge^2 T\mathcal{G})$; we say that $(\mathcal{G}, \pi)$ is a *Poisson groupoid* if the graph Γ_m of the multiplication map is coisotropic in $(\mathcal{G}, \pi) \times (\mathcal{G}, \pi) \times (\mathcal{G}, -\pi)$.

Examples...

What is the infinitesimal description of a Poisson groupoid? Need first infinitesimal description of Lie groupoids... Let us start with Poisson Lie groups.

3. Infinitesimal descriptions

3.1. **Lie bialgebras.** Recall different viewpoints to Lie algebras: Gerstenhaber bracket, CE-differential, Linear Poisson structures...

Given a Poisson Lie group (G, π), what is its description in terms of Lie algebra data?

Introduce Lie bialgebras; correspondence theorem for Poisson Lie groups and Lie bialgebras.

Examples.

Drinfeld doubles and Manin triples.

Examples of dual groups.

3.2. **Lie algebroids.** We now move to the infinitesimal description of a Lie groupoid.

A *Lie algebroid* over a manifold M is a vector bundle $A \to M$ equipped with a bundle map $\rho_A : A \to TM$ and a Lie bracket $[\cdot, \cdot]_A$ on the space of sections $\Gamma(A)$

such that

$$[u, fv]_A = f[u, v]_A + (\mathcal{L}_{\rho_{A(u)}} f)v,$$

for $u, v \in \Gamma(A)$ and $f \in C^\infty(M)$. This last condition is known as the *Leibniz identity.* The map $\rho : A \to TM$ is called the *anchor.*

Example 3.1.

- *Lie algebras* $(\mathfrak{g}, [\cdot, \cdot])$ *are Lie algebroids over a point.*
- $A = TM$, $[\cdot, \cdot]$ *is the Lie bracket of vector fields,* $\rho = \mathrm{id}$. *More generally, any involutive subbundle* $F \subseteq TM$ *defines a Lie algebroid.*
- *Lie algebroid structures on trivial line bundle* $\mathbb{R} \times M \to M$ *are the same as vector fields on* M.
- *Generalizing the previous item: given an infinitesimal action* $\rho : \mathfrak{g} \to \mathfrak{X}(M)$, *i.e, a Lie-algebra map, one can endow the trivial bundle* $A = M \times \mathfrak{g} \to M$ *with a natural Lie-algebroid structure. The anchor is given by the action map* $\mathfrak{g} \times M \to TM$, *and the bracket on* $\Gamma(A) = C^\infty(M, \mathfrak{g})$ *is uniquely defined by the Leibniz rule and the fact that it agrees with the Lie-algebra bracket on constant sections* $\mathfrak{g} \subset C^\infty(M, \mathfrak{g}$. *This Lie algebroid is called an* action Lie algebroid, *and denoted by* $\mathfrak{g} \times M$.

The kernel of the anchor map $\rho_A : A \to TM$ at each point defines a Lie algebra, and its image $\rho_A(A) \subset TM$ defines an integrable (generalized) smooth distribution on M. We refer to the integral leaves as *orbits* of the Lie algebroid.

Poisson manifolds give natural examples of Lie algebroids:

Example 3.2. *Let* (M, π) *be a Poisson manifold. Then* $T^*M \to M$ *is naturally a Lie algebroid, with anchor* $\pi^\sharp : T^*M \to TM$ *and bracket on* $\Omega^1(M)$ *defined by* $[df, dg] = d\{f, g\}$*; the general formula is*

$$[\alpha, \beta] = \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - d\pi(\alpha, \beta).$$

The orbits are the symplectic leaves, and the transversal Lie algebra structures agree with the ones coming from linearized Poisson structure on the conormal bundle.

Just as Lie algebroids, there are equivalent viewpoints to Lie algebroids that we will be useful to have in mind. A Lie algebroid $(A, [\cdot, \cdot], \rho)$ can be alternatively viewed in one of the following ways: as Gerstenhaber bracket on $\Gamma(\wedge^\bullet A)$, differential on $\Gamma(\wedge^\bullet A^*)$, or linear Poisson structure on A^*. In particular, Lie algebroids can be seen as particular types of Poisson manifolds.

Tangent and cotangent Lie algebroids...

The Lie algebroid of a Lie groupoid $\mathcal{G} \rightrightarrows M$: $A = \ker(T\mathsf{s})|_M \to M$, anchor is $\rho = T\mathsf{t}|_A : A \to TM$, Lie bracket is given by identifying $\Gamma(A)$ with right-invariant vector fields on $\mathcal{G}$.

Cotangent Lie groupoid: $T^*\mathcal{G} \rightrightarrows A^*$.

Lie theorems.

3.3. **Poisson structures as infinitesimal structures.** If $(\mathcal{G}, \omega) \rightrightarrows (M, \pi)$ is a symplectic groupoid, then the Lie algebroid A of $\mathcal{G}$ and the Lie algebroid T^*M of (M, π) are naturally isomorphic. Hence finding a symplectic groupoid for a Poisson manifold solves the integration problem for its Lie algebroid.

The converse is not exactly true; i.e., one may have Lie groupoids integrating the Lie algebroid T^*M which are not symplectic groupoids. But the source-simply-connected integration is always a symplectic groupoid integrating π. So if T^*M is integrable as a Lie algebroid, then a symplectic groupoid exists.

State the Poisson manifold/symplectic groupoid correspondence theorem.

Infinite-dimensional and finite-dimensional approaches to the proof.

3.4. **Lie bialgebroids.** Define Lie bialgebroids. Poisson groupoid/Lie bialgebroid correspondence theorem.

Doubles of Lie bialgebroids: Courant algebroids. Equivalence of Lie bialgebroids and pair of transversal Dirac structures.

4. Multiplicative structures

What about more general multiplicative objects? Just like multiplicative symplectic forms are related to Poisson structures, more general multiplicative forms (higher degree, nonclosed..), multivectors, tensors have their corresponding infinitesimal geometry.

4.1. **Multiplicative forms.** Definition of multiplicative forms. Infinitesimal description: the global-infinitesimal correspondence.

Examples: (twisted) Dirac structures and presymplctic groupoids, equivariant cohomology, Cartan-Dirac structures on Lie groups and AMM double.

4.2. **Multiplicative multivector fields.** The global-infinitesimal correspondence.

4.3. **Other multiplicative tensors.** Time permitting: General multiplicative tensors of mixed type. Infinitesimal description, global-infinitesimal correspondence.

Example: The global-infinitesimal correspondence for multiplicative Nijenhuis operators.

4.4. **Conclusion.** Other multiplicative objects, e.g. distributions and foliations...

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