Cluster Algebras and Compatible Poisson Structures

Poisson 2012, Utrecht

July, 2012

Reference:

- Cluster algebras and Poisson Geometry, M.Gekhtman, M.Shapiro, A.Vainshtein, AMS Surveys and Monographs, 2010 and references therein
- http://www.math.lsa.umich.edu/~fomin/cluster.html

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Note that the number of all minors grows exponentially with size. However, one can select (not uniquely) a family F of just n^2 minors of A such that A is totally positive iff every minor in the family is positive. (Berenshtein-Fomin-Zelevinsky)

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For $n = 3$ (total # of minors is 20),

$$F_1 = \{\Delta_3^3, \underline{\Delta_{23}^{23}}, \Delta_{23}^{13}, \Delta_{13}^{23}; \Delta_1^3, \Delta_3^1, \Delta_{12}^{23}, \Delta_{23}^{12}, \Delta_{123}^{123}\}$$

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- are connected by

$$\Delta_{13}^{13}\Delta_{23}^{23} = \Delta_{13}^{23}\Delta_{23}^{13} + \Delta_3^3\Delta_{123}^{123}$$

Properties

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- The intersection of opposite big Bruhat cells

$$B_+ w_0 B_+ \cap B_- w_0 B_- \subset GL(3)$$

coincides with

$$\{A \in GL(3) | \Delta_1^3 \Delta_3^1 \Delta_{12}^{23} \Delta_{23}^{12} \Delta_{123}^{123} \neq 0\}$$

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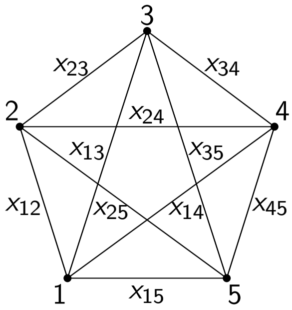
Homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}_{2,n+3}]$

$\mathrm{Gr}_{2,n+3} = \{V \subset \mathbb{C}^{n+3} : \dim(V) = 2\}$. The ring $\mathcal{A} = \mathbb{C}[\mathrm{Gr}_{2,n+3}]$ is generated by the Plücker coordinates x_{ij}, for $1 \le i < j \le n+3$.
Relations: $x_{ik}x_{jl} = x_{ij}x_{kl} + x_{il}x_{jk}$, for $i<j<k<l$.



sides: scalars

diagonals:
cluster variables

relations: "flips"

clusters:
triangulations

Each cluster has exactly n elements, so $\mathcal{A}$ is a cluster algebra of rank n.
The monomials involving "non-crossing" variables form a linear basis in $\mathcal{A}$ (studied in [Kung-Rota]).

Double Bruhat cell in $SL(3)$

$\mathcal{A} = \mathbb{C}[G^{u,v}]$, where $G^{u,v} = BuB \cap B_-vB_- =$

$$= \left\{ \begin{bmatrix} x & \alpha & 0 \\ \gamma & y & \beta \\ 0 & \delta & z \end{bmatrix} \in SL_3(\mathbb{C}) \ : \ \begin{matrix} \alpha \neq 0 & \beta \neq 0 \\ \gamma \neq 0 & \delta \neq 0 \end{matrix} \right\}$$

is a double Bruhat cell ($u,v \in \mathcal{S}_3$, $\ell(u) = \ell(v) = 2$).
Ground ring: $\mathbb{A} = \mathbb{C}[\alpha^{\pm 1}, \beta^{\pm 1}, \gamma^{\pm 1}, \delta^{\pm 1}]$.

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Five *cluster variables*. Exchange relations:

$$xy = \left\| \begin{matrix} x & \alpha \\ \gamma & y \end{matrix} \right\| + \alpha\gamma \qquad yz = \left\| \begin{matrix} y & \beta \\ \delta & z \end{matrix} \right\| + \beta\delta$$

$$x \left\| \begin{matrix} y & \beta \\ \delta & z \end{matrix} \right\| = \alpha\gamma z + 1 \qquad z \left\| \begin{matrix} x & \alpha \\ \gamma & y \end{matrix} \right\| = \beta\delta x + 1$$

$$\left\| \begin{matrix} x & \alpha \\ \gamma & y \end{matrix} \right\| \cdot \left\| \begin{matrix} y & \beta \\ \delta & z \end{matrix} \right\| = \alpha\beta\gamma\delta + y.$$

Definition

Exchange graph of a cluster algebra: vertices $\simeq$ clusters
edges $\simeq$ exchanges.

$\mathbb{T}_m$ *m-regular tree* with $\{1, 2, \dots m\}$-labeled edges,

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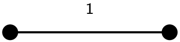

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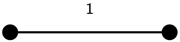
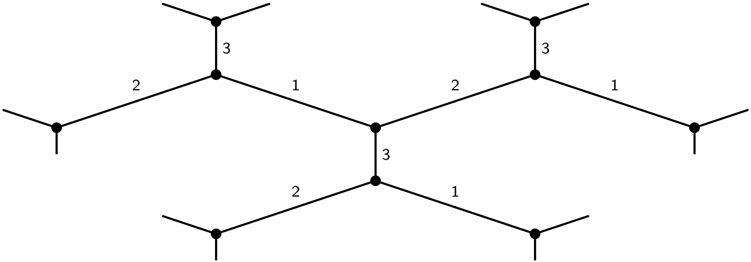


$\mathbb{T}_2$

$\mathbb{T}_3$



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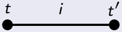
- variables $z_{m+1} = y_1, \dots, z_n = y_{n-m}$ are not affected by T_i.
- both $B(t)$ and $\mathbf{z}(t)$ are subject to cluster transformations defined as follows.

Cluster transformations

Cluster transformations

Cluster change

For an edge of $\mathbb{T}_m$ 

$i \in [1, \ldots, m]$

$T_i : \mathbf{z}(t) \mapsto \mathbf{z}(t')$ is defined as

$$\mathbf{x}_i(t') = \frac{1}{\mathbf{x}_i(t)} \left(\prod_{b_{ik}(t)>0} z_k(t)^{b_{ik}(t)} + \prod_{b_{ik}(t)<0} z_k(t)^{-b_{ik}(t)} \right)$$

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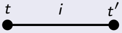
For an edge of $\mathbb{T}_m$ 

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Matrix mutation $B(t') = T_i(B(t))$,

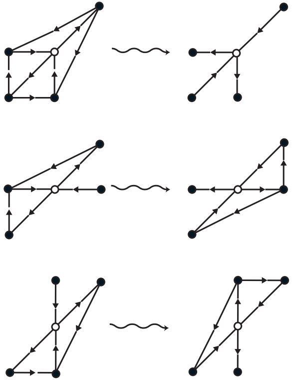
$$b_{kl}(t') = \begin{cases} -b_{kl}(t), & \text{if } (k-i)(l-i) = 0 \\ b_{kl}(t) + \dfrac{|b_{ki}(t)|b_{il}(t) + b_{ki}(t)|b_{il}(t)|}{2}, & \text{otherwise.} \end{cases}$$

Definition

Given some initial cluster t_0 put $z_i = z_i(t_0)$, $B = B(t_0)$. The cluster algebra $\mathcal{A}$ (or, $\mathcal{A}(B)$) is the subalgebra of the field of rational functions in cluster variables $z_1, \dots, z_n$ generated by the union of all cluster variables $z_i(t)$.

Examples of $T_i(B)$

A matrix $B(t)$ can be represented by a (weighted, oriented) graph.



The Laurent phenomenon

Theorem (FZ) In a cluster algebra, any cluster variable is expressed in terms of initial cluster as a Laurent polynomial.

Positivity Conjecture

All these Laurent polynomials have positive integer coefficients

Examples of Cluster Transformations

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- Short Plücker relation in $G_k(n)$

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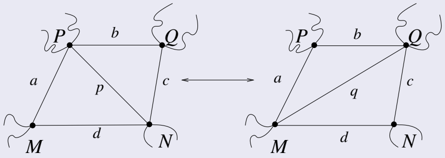
- Whitehead moves and Ptolemy relations in Decorated Teichmüller space:



$$f(p)f(q) = f(a)f(c) + f(b)f(d)$$

τ-coordinates

Nondegenerate coordinate change:

$$\tau_i(t) = \begin{cases} \prod_{j\neq i} z_j(t)^{b_{ij}(t)} & \text{for } i \leqslant m, \\ \prod_{j\neq i} z_j(t)^{b_{ij}(t)}/z_i(t) & \text{for } m+1 \leqslant i \leqslant n. \end{cases}$$

Exchange in direction i:

$$\tau_i \mapsto \frac{1}{\tau_i}; \qquad \tau_j \mapsto \begin{cases} \tau_j(1+\tau_i)^{b_{ij}}, & \text{if } b_{ij} > 0, \\ \tau_j \left(\frac{\tau_i}{1+\tau_i}\right)^{-b_{ij}}, & \text{otherwise.} \end{cases}$$

Definition

We say that a skew-symmetrizable matrix A is *reducible* if there exists a permutation matrix P such that PAP^T is a block-diagonal matrix, and *irreducible* otherwise. The *reducibility* $\rho(A)$ is defined as the maximal number of diagonal blocks in PAP^T. The partition into blocks defines an obvious equivalence relation $\sim$ on the rows (or columns) of A.

Compatible Poisson structures

A Poisson bracket $\{\cdot,\cdot\}$ is *compatible* with the cluster algebra $\mathcal{A}$ if, for any extended cluster $\widetilde{\mathbf{z}} = (z_1, \ldots, z_n)$

$$\{z_i, z_j\} = \omega_{ij} z_i z_j \ ,$$

where $\omega_{ij} \in \mathbb{Z}$ are constants for all $i, j \in [1, n+m]$.

Theorem

For an $B \in \mathbb{Z}_{n,n+m}$ as above of rank n the set of compatible Poisson brackets has dimension $\rho(B) + \binom{m}{2}$. Moreover, the coefficient matrices Ω^τ of these Poisson brackets in the basis τ are characterized by the equation $\Omega^\tau[m, n] = \Lambda B$ for some diagonal matrix $\Lambda =$ diagonal$(\lambda_1, \ldots, \lambda_n)$ where $\lambda_i = \lambda_j$ whenever $i \sim j$.

Degenerate exchange matrix

Example

Cluster algebra of rank 3 with trivial coefficients. Exchange matrix $B = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$. Compatible Poisson bracket must satisfy

$\{x_1, x_2\} = \lambda x_1 x_2,\ \{x_1, x_3\} = \mu x_1 x_3,\ \{x_2, x_3\} = \nu x_2 x_3$

Exercise: Check that these conditions imply $\lambda = \mu = \nu = 0$.

Conclusion: Only trivial Poisson structure is compatible with the cluster algebra.

What to do?

We will use the dual language of 2-forms

Compatible 2-forms

Definition

2-form ω is compatible with a collecion of functions $\{f_i\}$ if $\omega = \sum_{i,j} \omega_{ij} \frac{df_i}{f_i} \wedge \frac{df_j}{f_j}$

Definition

2-form ω is compatible with a cluster algebra if it is compatible with all clusters.

Exercise

Check that the form $\omega = \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} - \frac{dx_1}{x_1} \wedge \frac{dx_3}{x_3} + \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3}$ is compatible with the example above.

Compatible 2-forms

Theorem

For an $B \in \mathbb{Z}_{n,n+m}$ the set of Poisson brackets for which all extended clusters in $\mathfrak{A}(B)$ are log-canonical has dimension $\rho(B) + \binom{m}{2}$. Moreover, the coefficient matrices $\Omega^{\mathbf{x}}$ of these 2-forms in initial cluster are characterized by the equation $\Omega^{\mathbf{x}}[m,n] = \Lambda B$, where $\Lambda = diagonal(\lambda_1, \ldots, \lambda_n)$, with $\lambda_i = \lambda_j \neq 0$ whenever $i \sim j$.

Cluster manifold

For an abstract cluster algebra of geometric type $\mathcal{A}$ of rank m we construct an algebraic variety $\mathfrak{A}$ (which we call cluster manifold)

Idea: $\mathfrak{A}$ is a "good" part of $Spec(\mathcal{A})$.

We will describe $\mathfrak{A}$ by means of charts and transition functions.
For each cluster t we define an open chart

$$\mathfrak{A}(t) = Spec(\mathbb{C}[\mathbf{x}(t), \mathbf{x}(t)^{-1}, \mathbf{y}]),$$

where $\mathbf{x}(t)^{-1}$ means $x_1(t)^{-1}, \ldots, x_m(t)^{-1}$.
Transitions between charts are defined by exchange relations

$$x_i(t')x_i(t) = \prod_{b_{ik}(t)>0} z_k(t)^{b_{ik}(t)} + \prod_{b_{ik}(t)<0} z_k(t)^{-b_{ik}(t)}$$

$$z_j(t') = z_j(t) \quad j \neq i,$$

Finally, $\mathfrak{A} = \cup_t \mathfrak{A}(t)$.

Nonsingularity of $\mathfrak{A}$

$\mathfrak{A}$ contains only such points $p \in Spec(\mathcal{A})$ that there is a cluster t whose cluster elements form a coordinate system in some neighborhood of p.

Observation The cluster manifold $\mathfrak{A}$ is nonsingular and possesses a Poisson bracket that is log-canonical w.r.t. any extended cluster.

Let ω be one of these Poisson brackets.

Casimir of ω is a function that is in involution with all the other functions on $\mathfrak{A}$. All rational casimirs form a subfield F_C in the field of rational functions $\mathbb{C}(\mathfrak{A})$. The following proposition provides a complete description of F_C.

Lemma $F_C = F(\mathbf{m}_1, \ldots, \mathbf{m}_s)$, where $\mathbf{m}_j = \prod y_i^{\alpha_{ji}}$ for some integral α_{ji}, and $s = \text{corank}\omega$.

Toric action

We define a local toric action on the extended cluster t as the $\mathbb{C}^*$-action given by the formula $z_i(t) \mapsto z_i(t) \cdot \xi^{w_i(t)}$, $\xi \in \mathbb{C}^*$ for some integral $w_i(t)$ (called weights of toric action).
Local toric actions are compatible if taken in all clusters they define a global action on $\mathcal{A}$. This toric action is said to be an extension of the above local actions.
$\mathfrak{A}^0$ is the regular locus for all compatible toric actions on $\mathfrak{A}$.
$\mathfrak{A}^0$ is given by inequalities $y_i \neq 0$.

Symplectic leaves

$\mathfrak{A}$ is foliated into a disjoint union of symplectic leaves of ω.
Given generators $q_1, \dots, q_s$ of the field of rational casimirs F_C we have a map $Q : \mathfrak{A} \to \mathbb{C}^s$, $Q(x) = (q_1(x), \dots, q_s(x))$.
We say that a symplectic leaf $\mathcal{L}$ is generic if there exist s vector fields u_i on $\mathfrak{A}$ such that
a) at every point $x \in \mathcal{L}$, the vector $u_i(x)$ is transversal to the surface $Q^{-1}(Q(\mathcal{L}))$;
b) the translation along u_i for a sufficiently small time t gives a diffeomorphism between $\mathcal{L}$ and a close symplectic leaf $\mathcal{L}_t$.
Lemma $\mathfrak{A}^0$ is foliated into a disjoint union of generic symplectic leaves of the Poisson bracket ω.
Remark Generally speaking, $\mathfrak{A}^0$ does not coincide with the union of all "generic" symplectic leaves in $\mathfrak{A}$.

Connected components of $\mathfrak{A}^0$

Question: find the number $\#(\mathfrak{A}^0)$ of connected components of $\mathfrak{A}^0$.
Let $\mathcal{F}_2^n$ be an n-dimensional vector space over $\mathcal{F}_2$ with a fixed basis $\{e_i\}$.
Let B' be a $n \times n$- matrix with $\mathbb{Z}_2$ entries defined by the relation $B' \equiv B(t)\,(mod\ 2)$ for some cluster t, and let $\omega = \omega_t$ be a (skew-)symmetric bilinear form on $\mathcal{F}_2^n$, such that $\omega(e_i, e_j) = b'_{ij}$. Define a linear operator $\mathfrak{t}_i : \mathcal{F}_2^n \to \mathcal{F}_2^n$ by the formula $\mathfrak{t}_i(\theta) = \xi - \omega(\theta, e_i)e_i$, and let $\Gamma = \Gamma_t$ be the group generated by $\mathfrak{t}_i$, $1 \leqslant i \leqslant m$.
Theorem The number of connected components $\#(\mathfrak{A}^0)$ equals to the number of Γ_t-orbits in $\mathcal{F}_2^n$.

Cluster $\mathfrak{A}-$ and $\mathfrak{X}-$ manifolds

Coordinate ring	**Manifold**
$\mathcal{A}$ – cluster algebra with cluster coordinates x_i	$\mathfrak{A}$ – cluster variety with compatible 2-form
$\uparrow \pi^*$	$\pi : \tau_i = \prod_{j=1}^{n+m} x_j^{b_{ij}} \downarrow$
$\mathcal{X}$ – algebra generated by τ_i	$\mathfrak{X}$ – Poisson cluster variety

Let Ω be a 2-form.

$Ker\Omega = \{\text{vector } \xi : \Omega(\xi, \eta) = 0 \ \forall \eta\}$ provides a fibration of the underlying vector space.

$\{\text{space of fibers of } Ker\Omega\} \to \mathrm{Im}\, \pi$ is a local diffeomorphism.

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$Ker\Omega = \{\text{vector } \xi : \Omega(\xi, \eta) = 0 \ \forall \eta\}$ provides a fibration of the underlying vector space.

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More generally, let Ω be a compatible 2-form on a cluster manifold $\mathfrak{A}$ of coefficient-free cluster algebra $\mathcal{A}$.

$Ker\Omega$ determines an integrable distribution in $T\mathfrak{A}$.

Generic fibers of $Ker\Omega$ form a smooth manifold $\tilde{\mathfrak{X}}$ whose dimension is rank(B).

$\pi : \mathfrak{A} \to \tilde{\mathfrak{X}}$ is a natural projection.

Then, $\widetilde{\Omega} = \pi_*(\Omega)$ is a symplectic form on $\tilde{\mathfrak{X}}$ dual to the Poisson structure.

Application of Compatible Poisson Structures

If a Poisson variety $(\mathcal{M}, \{\cdot, \cdot\})$ possesses a coordinate chart that consists of regular functions whose logarithms have pairwise constant Poisson brackets, then one can use this chart to define a cluster algebra $\mathcal{A}_{\mathcal{M}}$ that is closely related (and, under rather mild conditions, isomorphic) to the ring of regular functions on $\mathcal{M}$ and such that $\{\cdot, \cdot\}$ is compatible with $\mathcal{A}_{\mathcal{M}}$.

Examples

- (Decorated) Teichmüller space has a natural structure of cluster algebra. Weyl-Petersson symplectic form is the unique symplectic form "compatible" with the structure of cluster algebra.
- There exists a cluster algebra structure on SL_n compatible with Sklyanin Poisson bracket. $\mathfrak{A}^0$ is the maximal double Bruhat cell.
- There exists a cluster algebra structure on Grassmanian compatible with push-forward of Sklyanin Poisson bracket. $\mathfrak{A}^0$ determined by the inequalities $\{$solid Plücker coordinate $\neq 0\}$.

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We will now provide a detailed discussion of the last two examples.

Poisson-Lie Groups

Let G be a Lie group.

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Then, if we have embedded Poisson subgroups B and B_- they define a Poisson-Lie structure on SL_2 they generate.

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$\{x_{ij}, x_{ki}\} = x_{ij}x_{ik}$ for $j < k$, $\{x_{ji}, x_{ki}\} = x_{ji}x_{ki}$ for $j < k$

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R-matrix

One can construct a Poisson-Lie bracket using *$R - matrix$*.

Definition

A map $R : g \to g$ is called a *classical $R - matrix$* if it satisfies modified Yang-Baxter equation

$$[R(\xi), R(\eta)] - R\left([R(\xi), \eta] + [\xi, R(\eta)]\right) = -[\xi, \eta]$$

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R-matrix Poisson bracket

R-matrix Poisson-Lie bracket on SL_n :

$$\{f_1, f_2\}(X) = \frac{1}{2}\left(\langle R(\nabla f_1(X)X), \nabla f_2(X)X\rangle - \langle R(X\nabla f_1(X)), X\nabla f_2(X))]\rangle\right),$$

where gradient $\nabla f \in sl_n$ defined w.r.t. trace form.

Example

For any matrix X we write its decomposition into a sum of lower triangular and strictly upper triangular matrices as

$$X = X_- + X_0 + X_+$$

The standard R-matrix $R : Mat_n \to Mat_n$ defined by

$$R(X) = X_+ - X_-$$

The standard R-matrix Poisson-Lie bracket:

$$\{x_{ij}, x_{\alpha\beta}\}(X) = \frac{1}{2}(sign(\alpha - i) + sign(\beta - j))x_{i\beta}x_{\alpha j}$$

Poisson Homogeneous Spaces

X is a homogeneous space of an algebraic group G, i.e.,

$$m : G \times X \to X.$$

G is equipped with Poisson-Lie structure.

Definition

Poisson bracket on X equips X with a structure of a Poisson homogeneous space if m is a Poisson map.

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To find a cluster structure in $G_k(n)$, need to find a coordinate system *compatible* with the bracket above.

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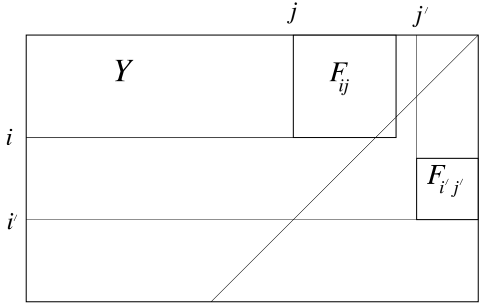

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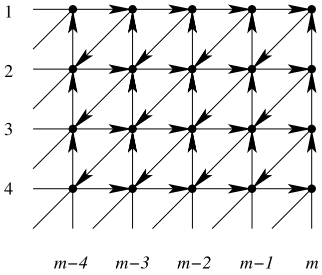

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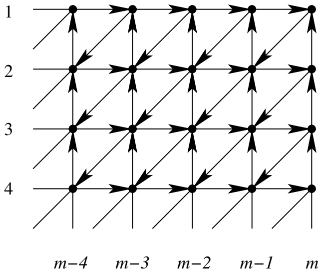

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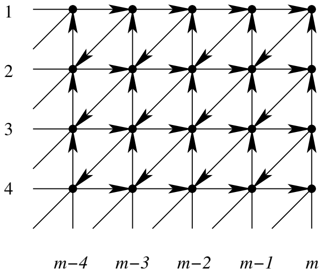

- Initial cluster transformations are built out of short Plücker relations.

And now for something completely different ....

Weighted network N in a disk

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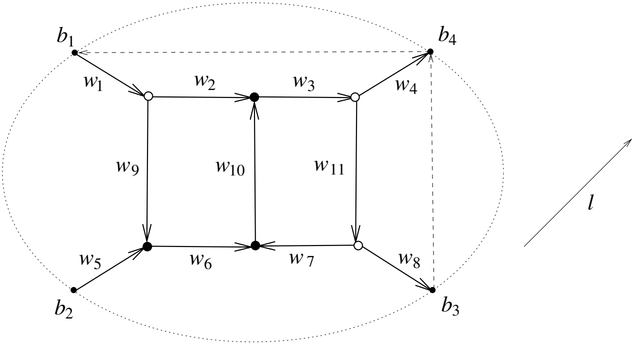
- To each $e \in E$ we assign a weight w_e.

Example



Boundary Measurements

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Paths and cycles

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For a closed oriented polygonal plane curve C

Boundary Measurements

Paths and cycles

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Example

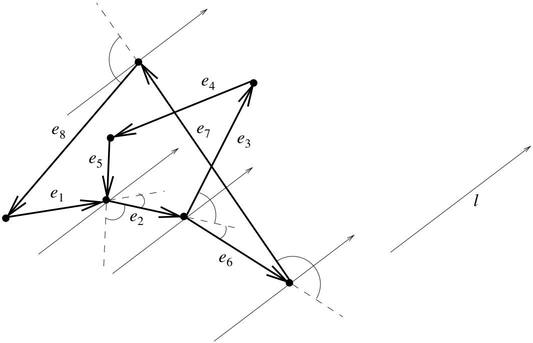

Figure: $c_l(e_1, e_2) = c_l(e_5, e_2) = 0$; $c_l(e_2, e_3) = 1$, $c_l(e_2, e_6) = 0$; $c_l(e_6, e_7) = 1$, $c_l(e_7, e_8) = 0$

Boundary Measurements (cont'd)

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A path P between a source b_i and a sink b_j

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Boundary Measurement Matrix: $M_N = (M(i_p, j_q))$

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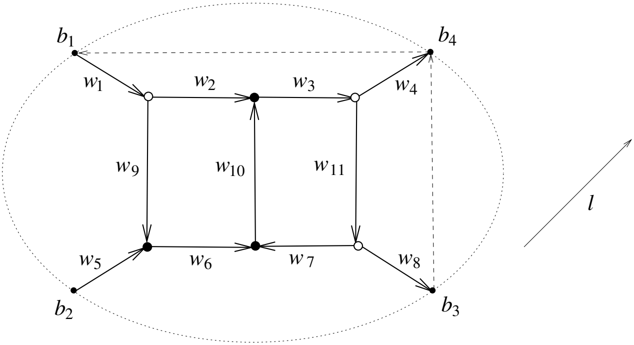

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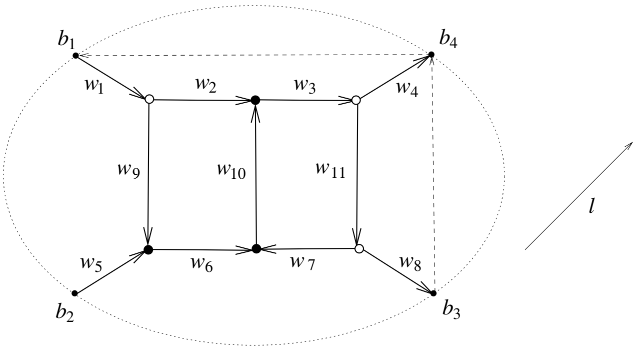

$$M_N = \begin{pmatrix} \dfrac{w_3 w_4 w_5 w_6 w_{10}}{1 + w_3 w_7 w_{10} w_{11}} & \dfrac{w_3 w_5 w_6 w_8 w_{11}}{1 + w_3 w_7 w_{10} w_{11}} \\ \dfrac{w_1 w_3 w_4 (w_2 + w_6 w_9 w_{10})}{1 + w_3 w_7 w_{10} w_{11}} & \dfrac{w_1 w_3 w_8 w_{11} (w_2 + w_6 w_9 w_{10})}{1 + w_3 w_7 w_{10} w_{11}} \end{pmatrix}.$$

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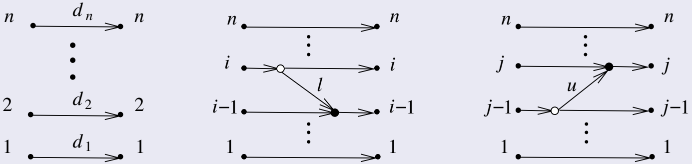

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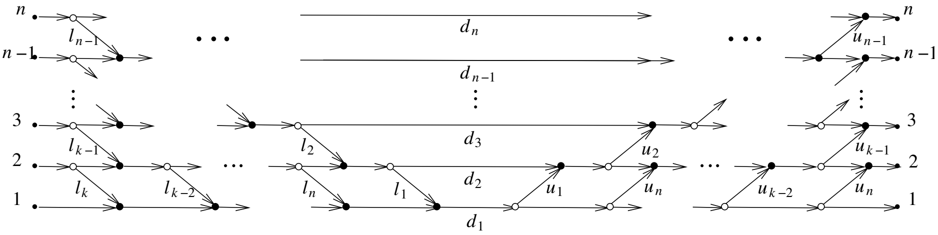

Figure: Generic planar network $\Longleftrightarrow$ Generic matrix

Restriction of $\{\cdot,\cdot\}_{R_0}$ to subgroups

$$B_+^{(i)} = \left\{ \mathbf{1}_{i-1} \oplus \begin{pmatrix} d & c \\ 0 & d^{-1} \end{pmatrix} \oplus \mathbf{1}_{n-i-1} \right\}, B(i)_- = \left\{ \mathbf{1}_{i-1} \oplus \begin{pmatrix} d & 0 \\ c & d^{-1} \end{pmatrix} \oplus \mathbf{1}_{n-i-1} \right\}$$

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Standard Poisson-Lie Structure via building blocks

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Can be described in terms of adjacent edges in corresponding networks !

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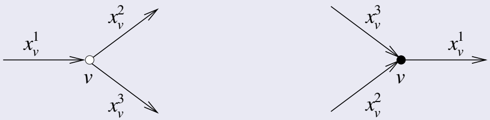

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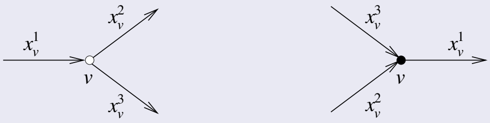

- Equip each $\mathbb{R}^3_v$ with a Poisson bracket $\rightsquigarrow \mathbb{C} = \oplus_v \mathbb{R}^3_v$ inherits $\{\cdot,\cdot\}_{\mathbb{C}} = \oplus_v \{\cdot,\cdot\}_v$.

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Proposition

Universal Poisson brackets $\{\cdot,\cdot\}_{\mathbb{C}}$ a 6-parametric family defined by relations

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at each white vertex v and

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Poisson Properties of the Boundary Measurement Map

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Theorem

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1. For any network N in a square with n sources and n sinks and for any choice of α_{ij}, β_{ij} the map $A_N : \mathbb{R}^{Edges} \to \mathrm{Mat}_n$ is Poisson w. r. t. the Sklyanin bracket associated with the R-matrix

$$R_{\alpha,\beta} = \frac{\alpha - \beta}{2}(\pi_+ - \pi_-) + \frac{\alpha + \beta}{2} S\pi_0,$$

where $S(e_{jj}) = \sum_{i=1}^{k} \mathfrak{s}(j-i)e_{ii}, \quad j = 1, \dots, k.$

Cluster algebra structure on boundary measurements

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Transformations preserving boundary measurements

- Gauge group acts on the space of edge weights

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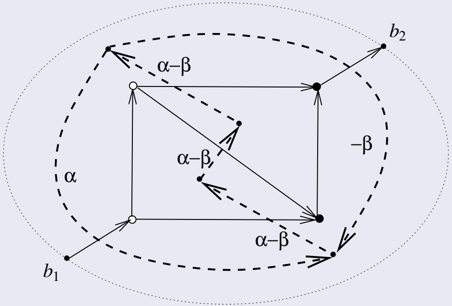

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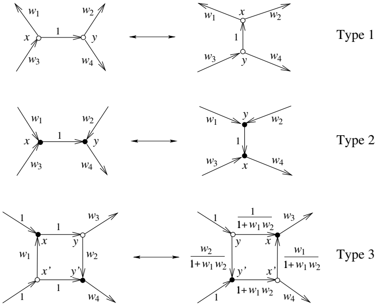

Figure: Elementary transformations

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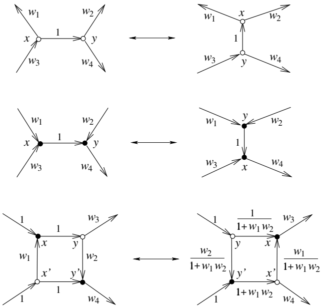

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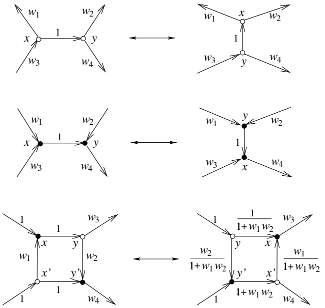

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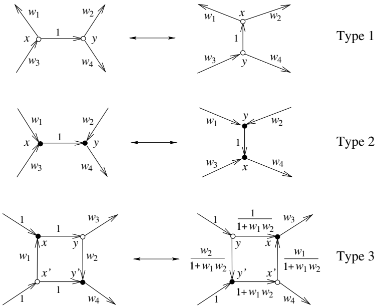

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3. In the case of only one source and one sink, both located at the same component of the boundary, the corresponding Poisson bracket is relevant in the study of Toda lattices and allows to construct a cluster algebra structure in the space of rational functions.

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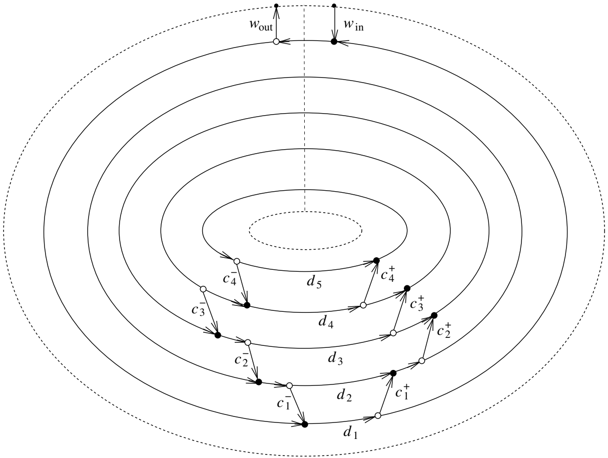

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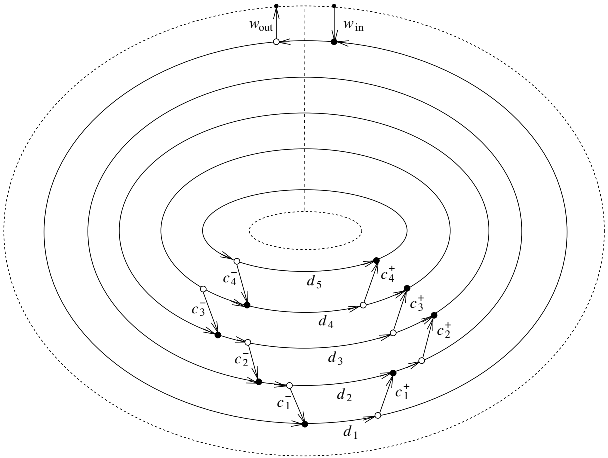

The graph "inside" represents a 5×5 matrix X.

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Theorem

Induced Poisson bracket on

$$\mathcal{R}_n = \left\{ M(\lambda) = \frac{Q(\lambda)}{P(\lambda)} : \deg P = n,\ \deg Q < n,\ P, Q \text{ are coprime} \right\}$$

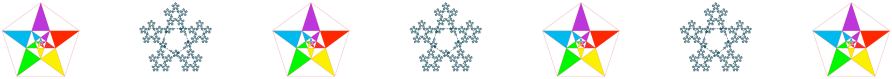
is

$$\{M(\lambda), M(\mu)\} = -(\lambda\ M(\lambda) - \mu\ M(\mu))\, \frac{M(\lambda) - M(\mu)}{\lambda - \mu}.$$

It coincides with the one induced by the quadratic Poisson structure for Toda flows.

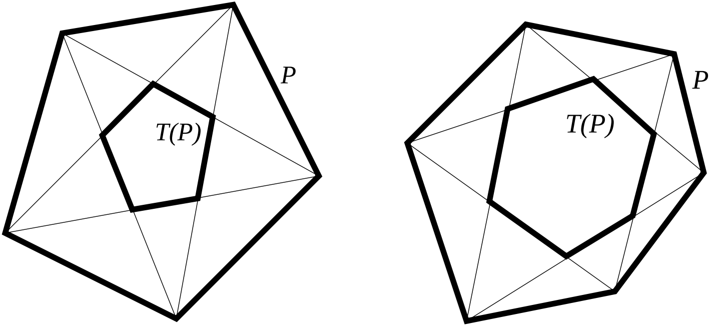
Now let's tie it all together with an example...

Pentagram map



R. Schwartz, V. Ovsienko, S. Tabachnikov, S. Morier-Genoud, M. Glick, F. Soloviev, B. Khesin, G. Mari-Beffa, M. Gekhtman, M. Shapiro, A. Vainshtein, V. Fock, A. Marshakov , ...

Pentagram Map T:



Acts on projective equivalence classes of closed n-gons (dim$= 2n - 8$)

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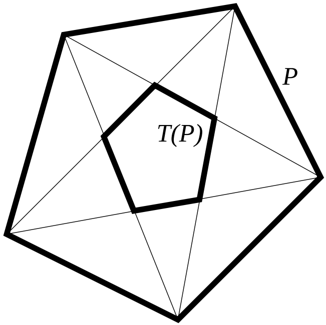
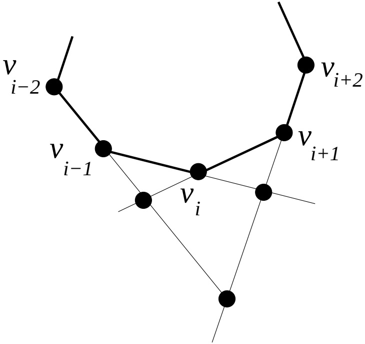


Acts on projective equivalence classes of closed n-gons (dim$= 2n - 8$) or *twisted* n-gons with monodromy M (dim$= 2n$).

Corner coordinates: left and right cross-ratios $X_1, Y_1, \ldots, X_n, Y_n$.



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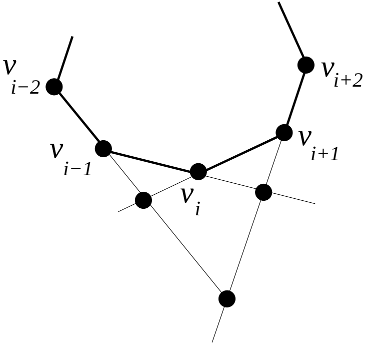

The map T becomes:

$$X_i^* = X_i \frac{1 - X_{i-1}\, Y_{i-1}}{1 - X_{i+1}\, Y_{i+1}}, \qquad Y_i^* = Y_{i+1} \frac{1 - X_{i+2}\, Y_{i+2}}{1 - X_i\, Y_i}.$$

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Complete integrability on the space of closed polygons has been proven as well:

F. Soloviev. *Integrability of the Pentagram Map*, arXiv:1106.3950;

V. Ovsienko, R. Schwartz, S. Tabachnikov. *Liouville-Arnold integrability of the pentagram map on closed polygons*, arXiv:1107.3633.

Cluster algebras and integrable systems:

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- Dimers (Goncharov, Kenyon)

Cluster interpretation for the Pentagram Map?

M. Glick. *The pentagram map and Y-patterns* (Adv. Math., **227** (2011), 1019--1045) :
Considered the dynamics in the $2n-1$-dimensional quotient space by the scaling symmetry $(X, Y) \mapsto (tX, t^{-1}Y)$:

$$p_i = -X_{i+1}Y_{i+1}, \quad q_i = -\frac{1}{Y_i X_{i+1}},$$

and proved that it was a Y-type cluster algebra dynamics.

Cluster dynamics

Recall - Matrix mutations and transformations of τ-coordinates in the definition of a cluster algebra is equivalent to quiver mutations:

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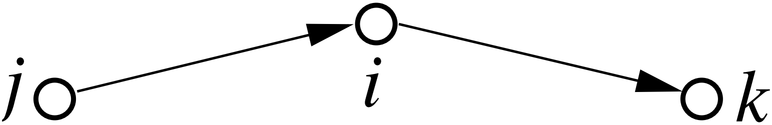
Recall - Matrix mutations and transformations of τ-coordinates in the definition of a cluster algebra is equivalent to quiver mutations: Given a *quiver* (an oriented graph with no loops or 2-cycles) whose vertices are labeled by variables τ_i (rational functions in some free variables), the mutation associated with a vertex i is



$$\tau_i^* = \frac{1}{\tau_i}, \quad \tau_j^* = \frac{\tau_j \tau_i}{1+\tau_i}, \quad \tau_k^* = \tau_k(1+\tau_i);$$

the rest of the variables are intact.

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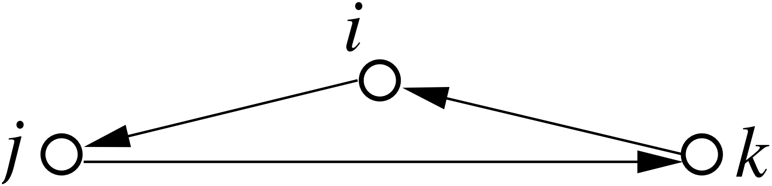

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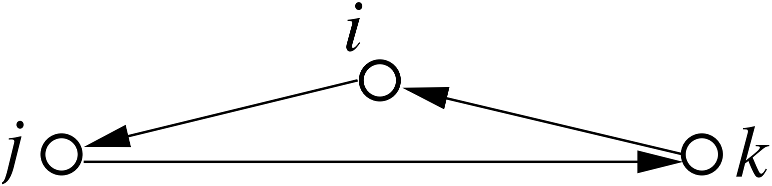
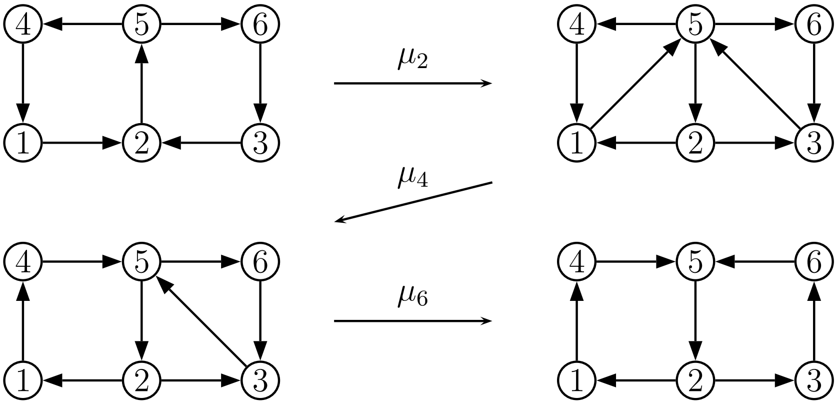
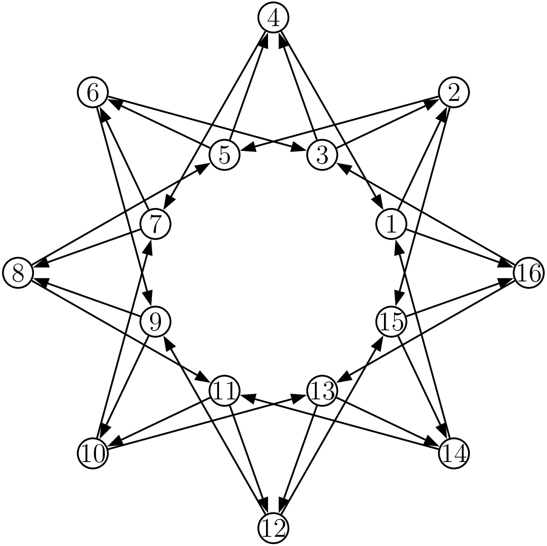


The mutation on a given vertex is an involution.

Example of mutations:



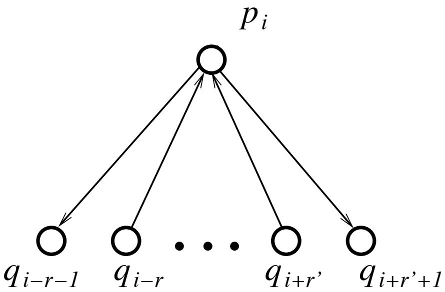
Glick's quiver ($n = 8$):



GSTV, ERA 19 (2012), 1-17:

Generalize Glick's quiver:

consider the homogeneous bipartite graph $\mathcal{Q}_{k,n}$ where $r = [k/2] - 1$, and $r' = r$ for k even and $r' = r + 1$ for k odd (each vertex is 4-valent):



Dynamics: mutations on all p-vertices, followed by swapping p and q; this is the map $\overline{T}_k$:

$$q_i^* = \frac{1}{p_i}, \quad p_i^* = q_i \frac{(1+p_{i-r-1})(1+p_{i+r+1})p_{i-r}p_{i+r}}{(1+p_{i-r})(1+p_{i+r})}, \quad k \text{ even},$$
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(The Pentagram Map corresponds to $\overline{T}_3$).

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- The Poisson bracket is compatible with the cluster algebra determined by the quiver.

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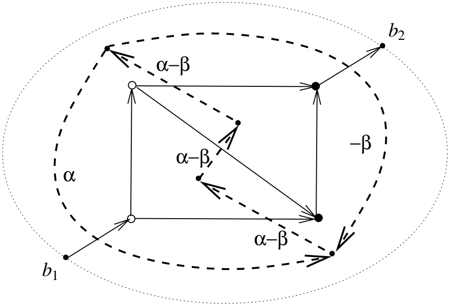
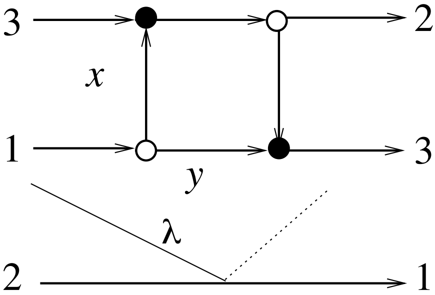


The Poisson bracket above can be realized as a universal Poisson bracket for the dual network.

Weighted directed networks on the cylinder and the torus

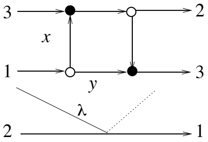
Example:



- Two kinds of vertices, white and black.
- Convention: an edge weight is 1, if not specified.
- The *cut* is used to introduce a *spectral parameter* λ.

Boundary measurements :

The network



corresponds to the matrix

$$\begin{pmatrix} 0 & x & x+y \\ \lambda & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Concatenation of networks $\mapsto$ product of matrices.

Gauge group: at a vertex, multiply the weights of the incoming edges and divide the weights of the outgoing ones by the same function. Leaves the boundary measurements intact.

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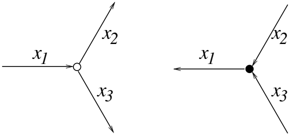
Poisson bracket (6-parameter): $\{x_i, x_j\} = c_{ij} x_i x_j, \; i \neq j \in \{1, 2, 3\}$



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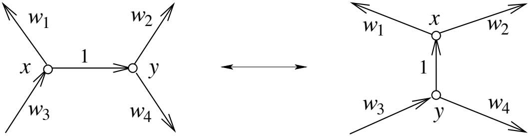
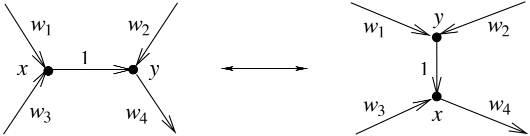
2. Universal Poisson brackets on edge weights lead to trigonometric R-matrix brackets in the case when sources and sinks are located at opposite ends of a cylinder:

$$\{M(\lambda) \overset{\otimes}{,} M(\mu)\} = [R(\lambda, \mu), M(\lambda) \otimes M(\mu)]$$

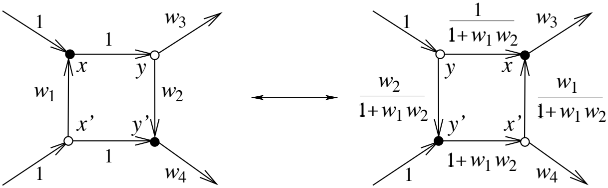
Postnikov moves (do not change the boundary measurements):



Type 1

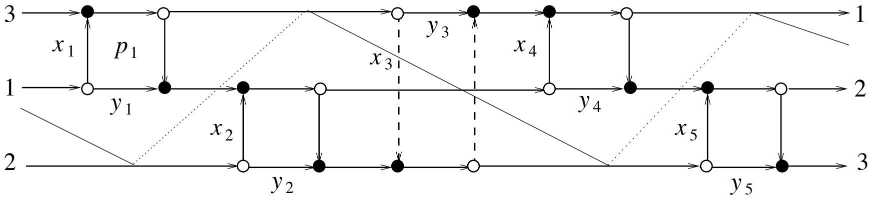


Type 2



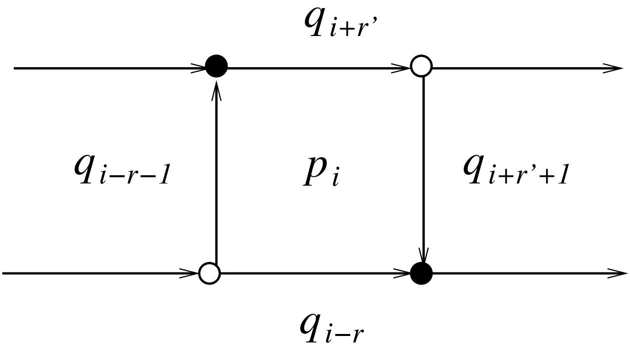
Type 3

Consider a network whose dual graph is the quiver $\mathcal{Q}_{k,n}$.
It is drawn on the torus. Example, $k = 3, n = 5$:



Convention: white vertices of the graph are on the left of oriented edges of the dual graph.

The network is made of the blocks:

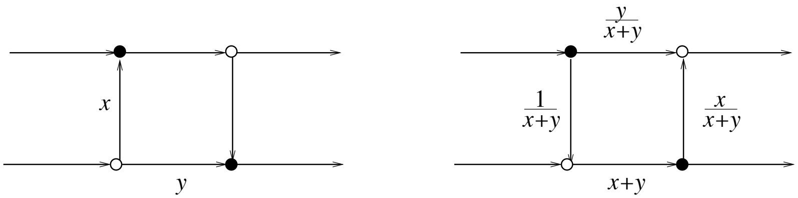


Face weights:

$$p_i = \frac{y_i}{x_i}, \quad q_i = \frac{x_{i+1+r}}{y_{i+r}}.$$

This is a projection $\pi : (x, y) \mapsto (p, q)$ with 1-dimensional fiber.

(x, y)-dynamics: mutation (Postnikov type 3 move on each p-face),



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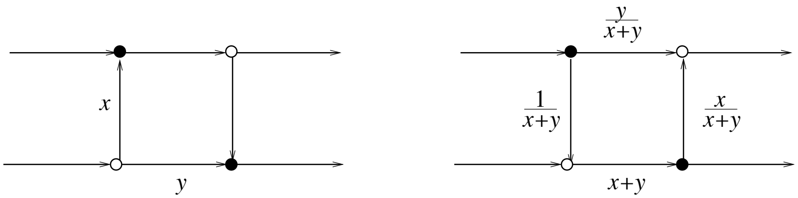

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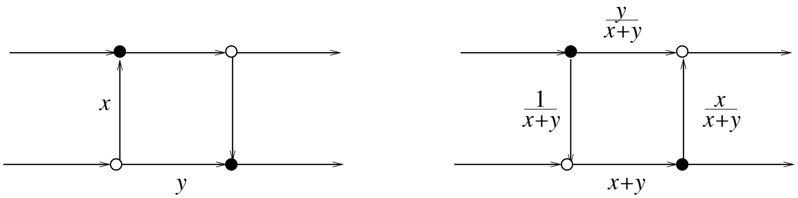
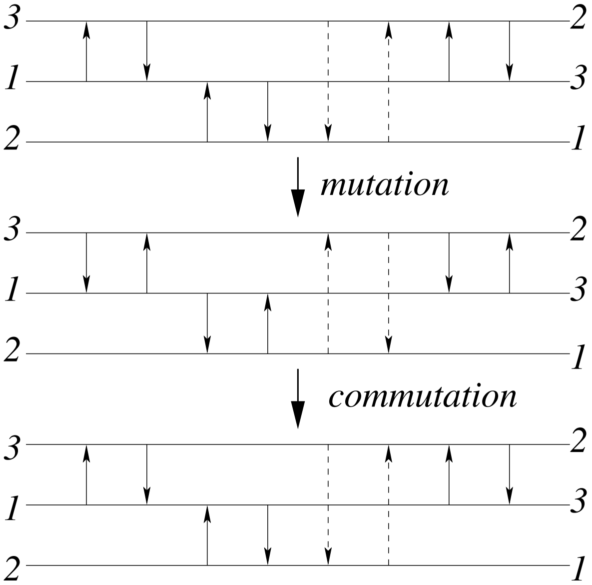


followed by the Postnikov type 1 and 2 moves on the white-white and black-black edge (this interchanges p- and q-faces), including moving across the vertical cut,and finally, re-calibration to restore 1s on the appropriate edges. These moves preserve the conjugacy class of the boundary measurement matrix.

Schematically:



This results in the map T_k:

$$x_i^* = x_{i-r-1}\frac{x_{i+r}+y_{i+r}}{x_{i-r-1}+y_{i-r-1}}, \quad y_i^* = y_{i-r}\frac{x_{i+r+1}+y_{i+r+1}}{x_{i-r}+y_{i-r}}, \quad k \text{ even},$$
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Relation with the pentagram map: the change of variables

$$x_i \mapsto Y_i, \quad y_i \mapsto -Y_i X_{i+1} Y_{i+1},$$

identifies T_3 with the pentagram map.

Complete integrability of the maps T_k

Main point: all ingredients are determined by the combinatorics of the network !

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- Invariant Poisson bracket (in the "stable range" $n \geq 2k-1$) :

$$\{x_i, x_{i+l}\} = -x_i x_{i+l}, 1 \leq l \leq k-2;\ \{y_i, y_{i+l}\} = -y_i y_{i+l}, 1 \leq l \leq k-1;$$
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the indices are cyclic.

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  the indices are cyclic.

- The functions $\prod x_i$ and $\prod y_i$ are Casimir. If n is even and k is odd, one has four Casimir functions:

$$\prod_{i \text{ even}} x_i, \quad \prod_{i \text{ odd}} x_i, \quad \prod_{i \text{ even}} y_i, \quad \prod_{i \text{ odd}} y_i.$$

Lax matrices, monodromy, integrals

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- For $k \geq 3$,

$$L_i = \begin{pmatrix} 0 & 0 & 0 & \dots & x_i & x_i + y_i \\ \lambda & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 1 \end{pmatrix},$$

and for $k = 2$,

$$L_i = \begin{pmatrix} \lambda x_i & x_i + y_i \\ \lambda & 1 \end{pmatrix}.$$

Lax matrices, monodromy, integrals

- For $k \geq 3$,

$$L_i = \begin{pmatrix} 0 & 0 & 0 & \dots & x_i & x_i + y_i \\ \lambda & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 1 \end{pmatrix},$$

  and for $k = 2$,

$$L_i = \begin{pmatrix} \lambda x_i & x_i + y_i \\ \lambda & 1 \end{pmatrix}.$$

- The boundary measurement matrix is $M(\lambda) = L_1 \cdots L_n$. The characteristic polynomial

$$\det(M(\lambda) - z) = \sum I_{ij}(x, y) z^i \lambda^j.$$

  is T_k-invariant. The integrals I_{ij} are in involution.

Zero curvature (Lax) representation:

$$L_i^* = P_i L_{i+r-1} P_{i+1}^{-1}$$

where L_i are the Lax matrices and

$$P_i = \begin{pmatrix} 0 & \frac{x_i}{\lambda\sigma_i} & \frac{y_{i+1}}{\lambda\sigma_{i+1}} & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{x_{i+1}}{\sigma_{i+1}} & \frac{y_{i+2}}{\sigma_{i+2}} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{x_{i+k-4}}{\sigma_{i+k-4}} & \frac{y_{i+k-3}}{\sigma_{i+k-3}} & 0 \\ -\frac{1}{\sigma_{i+k-2}} & 0 & 0 & \dots & 0 & \frac{x_{i+k-3}}{\sigma_{i+k-3}} & 1 \\ \frac{1}{\sigma_{i+k-2}} & -\frac{1}{\lambda\sigma_{i+k-1}} & 0 & \dots & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda\sigma_{i+k-1}} & 0 & \dots & 0 & 0 & 0 \end{pmatrix},$$

with $\sigma_i = x_i + y_i$.

Geometric interpretation

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- $\mathcal{P}^0_{k,n} \subset \mathcal{P}_{k,n}$ consists of the polygons with the following property: for every i, the vertices V_i, V_{i+1}, V_{i+k-1} and V_{i+k} span a projective plane. These are called *corrugated* polygons.

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- The consecutive $(k-1)$-diagonals of a corrugated polygon intersect. The resulting polygon is again corrugated. One gets a pentagram-like $k-1$-diagonal map on $\mathcal{P}^0_{k,n}$ (*higher pentagram map*). For $k=3$, this is the pentagram map.

- Coordinates: lift the vertices V_i of a corrugated polygon to vectors $\widetilde{V}_i$ in $\mathbb{R}^k$ so that the linear recurrence holds

$$\widetilde{V}_{i+k} = y_{i-1}\widetilde{V}_i + x_i\widetilde{V}_{i+1} + \widetilde{V}_{i+k-1},$$

where x_i and y_i are n-periodic sequences. These are coordinates in $\mathcal{P}^0_{k,n}$. In these coordinates, the map is identified with T_k.

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- The same functions x_i, y_i can be defined on polygons in the projective plane. One obtains integrals of the "deeper" diagonal maps on twisted polygons in $\mathbf{RP}^2$.

Case $k = 2$

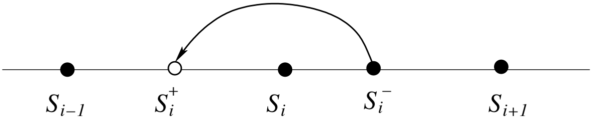
Consider the space $\mathcal{S}_n$ of pairs of twisted n-gons (S^-, S) in $\mathbf{RP}^1$ with the same monodromy. Consider the projectively invariant projection ϕ to the (x, y)-space (cross-ratios):

$$x_i = \frac{(S_{i+1} - S^-_{i+2})(S^-_i - S^-_{i+1})}{(S^-_i - S_{i+1})(S^-_{i+1} - S^-_{i+2})}$$

$$y_i = \frac{(S^-_{i+1} - S_{i+1})(S^-_{i+2} - S_{i+2})(S^-_i - S^-_{i+1})}{(S^-_{i+1} - S_{i+2})(S^-_i - S_{i+1})(S^-_{i+1} - S^-_{i+2})}.$$

Then x_i, y_i are coordinates in $\mathcal{S}_n / PGL(2, \mathbb{R})$.

Define a transformation $F_2(S^-, S) = (S, S^+)$, where S^+ is given by the following local leapfrog rule: given points $S_{i-1}, S_i^-, S_i, S_{i+1}$, the point S_i^+ is obtained by the reflection of S_i^- in S_i in the projective metric on the segment $[S_{i-1}, S_{i+1}]$:



The projection ϕ conjugates F_2 and T_2.

In formulas:

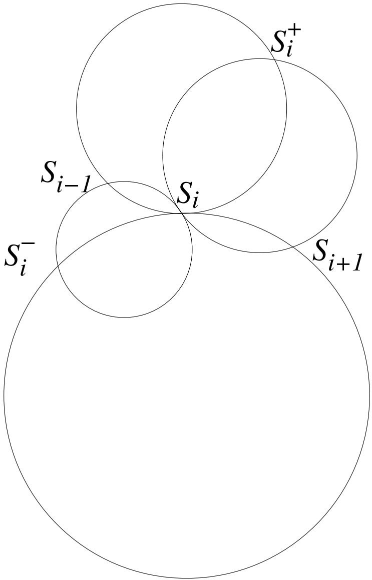
$$\frac{1}{S_i^+ - S_i} + \frac{1}{S_i^- - S_i} = \frac{1}{S_{i+1} - S_i} + \frac{1}{S_{i-1} - S_i},$$

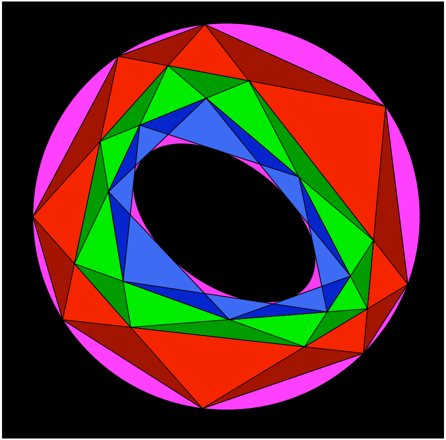
or, equivalently,

$$\frac{(S_i^+ - S_{i+1})(S_i - S_i^-)(S_i - S_{i-1})}{(S_i^+ - S_i)(S_{i+1} - S_i)(S_i^- - S_{i-1})} = -1,$$

(Toda-type equations).

In $\mathbf{CP}^1$, a circle pattern interpretation (generalized Schramm's pattern):





That's all, folks!