SKT Geometry

ArXiv:1203.0493

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Poisson Geometry in Mathematics and Physics
Utrecht 2012

Introduction — SKT structures

- *Kähler structure with torsion*: Hermitian structure (g, I) with a connection ∇ with $\mathrm{Tor}(\nabla) = H \in \Omega^3(M)$ such that

$$\nabla g = \nabla I = 0.$$

- Kähler structure with *strong torsion* (SKT): $dH = 0$.
- SKT structure: Hermitian structure (g, I) such that

$$dd^c\omega = 0.$$

- $H = d^c\omega$.

Examples

- **Kähler manifolds**;
- Compact even dimensional **Lie groups**;
- **Compact complex surfaces** (Gauduchon);
- **Instanton moduli space** over compact complex surfaces;
- **Instanton moduli space** over Hermitian manifolds with Gauduchon metrics;
- Classification on **6-nilmanifolds**;

Kähler vs. SKT

	Kähler	SKT
Decomposition of cohomology	✓	
Hodge theory	✓	
Frölicher spectral seq. degenerates	✓	
Formality	✓	
Unobstructed deformations	✓	

Fino, Parton and Salamon. Families of strong KT structures in six dimensions. *Comment. Math. Helv.* 2004. (arXiv:math/0209259)

Kähler vs. SKT

	Kähler	SKT
Decomposition of cohomology	✓	✗
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Frölicher spectral seq. degenerates	✓	✗
Formality	✓	✗
Unobstructed deformations	✓	✗

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Kähler vs. SKT

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Decomposition of cohomology	✓	✗
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Kähler vs. SKT

	Kähler	SKT
Decomposition of cohomology	✓	✓
Hodge theory	✓	✓
Frölicher spectral seq. degenerates	✓	✓
Formality	✓	✗
Unobstructed deformations	✗	✗

Kähler vs. GK vs. SKT

	Kähler	GK	SKT
Decomposition of cohomology	✓	✓	✓
Hodge theory	✓	✓	✓
Frölicher spectral seq. degenerates	✓	✓	✓
Formality	✓	✗	✗
Unobstructed deformations	✗	✗	✗

Main insights

- SKT structures are also generalized structures à la Hitchin.
- Description of intrinsic torsion of generalized almost Hermitian structures.

Outline of Topics

1. Geometry of $T \oplus T^*$
2. Nijenhuis tensor and intrinsic torsion
3. Hodge theory
4. Deformations

Geometry of $T \oplus T^*$ — Natural pairing

- **Natural pairing**

$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\eta(X)+\xi(Y)).$$

- Action of $T \oplus T^*$ on $\wedge^\bullet T^*$:

$$(X+\xi)\cdot\varphi = \iota_X\varphi + \xi\wedge\varphi.$$

- Extends to an action of $\mathrm{Clif}(T \oplus T^*)$ on $\wedge^\bullet T^*$:

$$v\cdot(v\cdot\varphi) = \langle v,v\rangle\varphi.$$

- $\wedge^\bullet T^* \rightsquigarrow$ spinors.
- Spin invariant pairing: $(\cdot,\cdot)_{Ch} : \wedge^\bullet T^* \otimes \wedge^\bullet T^* \longrightarrow \wedge^{top} T^*$.

Geometry of $T \oplus T^*$ — Generalized metric

- A **generalized metric** is an orthogonal, self-adjoint bundle isomorphism:
  $$\mathcal{G} : T \oplus T^* \longrightarrow T \oplus T^*;$$
  such that
  $$\langle \mathcal{G}v, v \rangle > 0.$$
- $\mathcal{G}^{-1} = \mathcal{G}^t = \mathcal{G} \quad \Rightarrow \quad \mathcal{G}^2 = \mathrm{Id}.$
- $\mathcal{G}$ is determined by its $+1$-eigenspace V_+.

Geometry of $T \oplus T^*$ — Generalized Hodge star

- Generalized metric + orientation $\Rightarrow$ generalized Hodge star $\star$
$$(\varphi, \star\varphi)_{Ch} > 0.$$
- $\star^2 = (-1)^{\frac{m(m-1)}{2}}$.
- SD forms = $-i^{\frac{m(m-1)}{2}}$-eigenspace;
  ASD forms = $i^{\frac{m(m-1)}{2}}$-eigenspace;

Geometry of $T \oplus T^*$ — Gen. almost complex structure

- Generalized almost complex structure:

$$\mathcal{J} : T \oplus T^* \longrightarrow T \oplus T^*; \qquad \mathcal{J}^2 = -\mathrm{Id};$$

  $\mathcal{J}$ is orthogonal.
- $\mathcal{J} \Leftrightarrow L \subset (T \oplus T^*) \otimes \mathbb{C}$, maximal isotropic $L \cap \overline{L} = \{0\}$.
- $\mathcal{J}^t = \mathcal{J}^{-1} = -\mathcal{J} \Rightarrow \mathcal{J} \in \wedge^2(T \oplus T^*) = \mathfrak{spin}(T \oplus T^*)$.
- $\mathcal{J}$ splits $\wedge^\bullet T^*$ into its ik-eigenspaces:

$$\wedge^\bullet T^*_{\mathbb{C}} M = \oplus_{-n \leq k \leq n} U^k.$$

Geometry of $T \oplus T^*$ — Gen. almost Hermitian str.

- Generalized almost Hermitian structure: $(\mathcal{G}, \mathcal{J}_1)$

$$\mathcal{G}\mathcal{J}_1 = \mathcal{J}_1\mathcal{G}.$$

- $\mathcal{J}_2 = \mathcal{G}\mathcal{J}_1$ is a gcs and $\mathcal{J}_2\mathcal{J}_1 = \mathcal{J}_1\mathcal{J}_2$.
- $(T \oplus T^*) \otimes \mathbb{C} = V_+^{1,0} \oplus V_+^{0,1} \oplus V_-^{1,0} \oplus V_-^{0,1}$.
- $\wedge^\bullet T^*_{\mathbb{C}}M = \oplus_{p,q} U^p_{\mathcal{J}_1} \cap U^q_{\mathcal{J}_2}$.

Geometry of $T \oplus T^*$ — Gen. almost Hermitian str.

$$U^{0,3}$$
$$U^{-1,2} \qquad U^{1,2}$$
$$U^{-2,1} \qquad U^{0,1} \qquad U^{2,1}$$
$$U^{-3,0} \qquad U^{-1,0} \qquad U^{1,0} \qquad U^{3,0}$$
$$U^{-2,-1} \qquad U^{0,-1} \qquad U^{2,-1}$$
$$U^{-1,-2} \qquad U^{1,-2}$$
$$U^{0,-3}$$

Spaces $U^{p,q}$ on a 6-dimensional generalized almost Hermitian structure.

Geometry of $T \oplus T^*$ — Gen. almost Hermitian str.

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- $\wedge^\bullet T^*_{\mathbb{C}}M = \oplus_{p,q} U^p_{\mathcal{J}_1} \cap U^q_{\mathcal{J}_2}$.
- $\star = -e^{\frac{\pi\mathcal{J}_1}{2}} e^{\frac{\pi\mathcal{J}_2}{2}}$

$$\star|_{U^{p,q}} = -i^{p+q}.$$

Geometry of $T \oplus T^*$ — Gen. almost Hermitian str.

$$
\begin{array}{ccccccc}
 & & & U^{0,3} & & & \\
 & & U^{-1,2} & & U^{1,2} & & \\
 & U^{-2,1} & & U^{0,1} & & U^{2,1} & \\
U^{-3,0} & & U^{-1,0} & & U^{1,0} & & U^{3,0} \\
 & U^{-2,-1} & & U^{0,-1} & & U^{2,-1} & \\
 & & U^{-1,-2} & & U^{1,-2} & & \\
 & & & U^{0,-3} & & &
\end{array}
$$

SD and **ASD** forms on a 6-dimensional generalized almost Hermitian structure.

Geometry of $T \oplus T^*$ — Gen. almost Hermitian str.

$$\begin{array}{ccccccc}
 & & & U^{0,3} & & & \\
 & & U^{-1,2} & & U^{1,2} & & \\
 & U^{-2,1} & & U^{0,1} & & U^{2,1} & \\
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 & U^{-2,-1} & & U^{0,-1} & & U^{2,-1} & \\
 & & U^{-1,-2} & & U^{1,-2} & & \\
 & & & U^{0,-3} & & &
\end{array}$$

SD and **ASD** forms on a 6-dimensional generalized almost Hermitian structure.

Geometry of $T \oplus T^*$ — Courant bracket

- Courant bracket

$$[\![X+\xi, Y+\eta]\!]_H = [X,Y] + \mathcal{L}_X\eta - \iota_Y d\xi - \iota_Y\iota_X H.$$

-

$$[\![v_1, v_2]\!]_H \cdot \varphi = \{\{v_1, d^H\}, v_2\} \cdot \varphi.$$

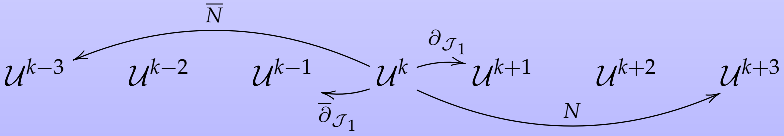
Nijenhuis tensor and intrinsic torsion

- Given a gacs $\mathcal{J}$, define

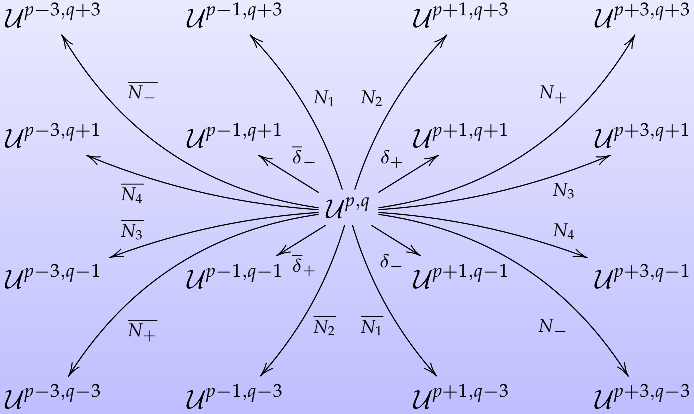
$$N : \Gamma(\overline{L}) \times \Gamma(\overline{L}) \times \Gamma(\overline{L}) \longrightarrow \Omega^0(M;\mathbb{C})$$

$$N(v_1, v_2, v_3) = -2\langle [\![v_1, v_2]\!], v_3\rangle.$$

- $\mathcal{J}$ is integrable iff $N \equiv 0$.
- $N \in \Gamma(\wedge^3 L)$.

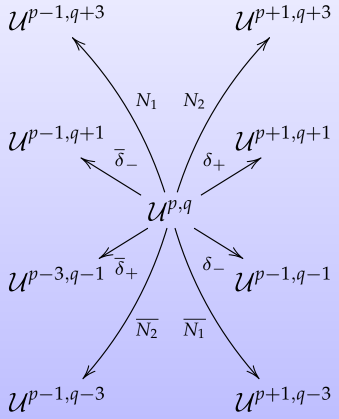


Nijenhuis tensor and intrinsic torsion



Components of d^H for a generalized almost Hermitian structure.

Nijenhuis tensor and intrinsic torsion



Components of d^H for a generalized Hermitian structure.

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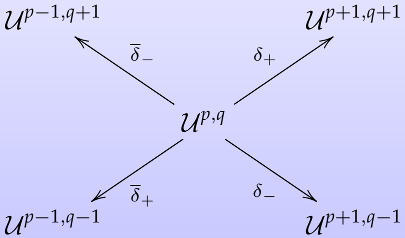


Components of d^H for a generalized Kähler.

Nijenhuis tensor and intrinsic torsion

Definition

The tensors N_α, $\alpha = 1, 2, 3, 4$ and $\pm$ are the components of the intrinsic torsion of a $U(n) \times U(n)$ structure.

Nijenhuis tensor and intrinsic torsion

Proposition (Gualtieri 04/Cavalcanti 06)

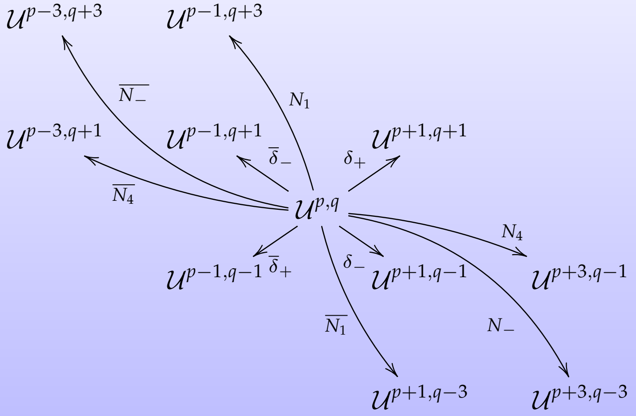
A **(positive) SKT structure** *is a generalized metric $\mathcal{G}$ and complex structure $\mathcal{I}_+$ on V_+ such that*

$$[\![\Gamma(V_+^{1,0}), \Gamma(V_+^{1,0})]\!] \subset \Gamma(V_+^{1,0}).$$

Similarly, a **(negative) SKT structure** *is a complex structure $\mathcal{I}_-$ on V_- such that*

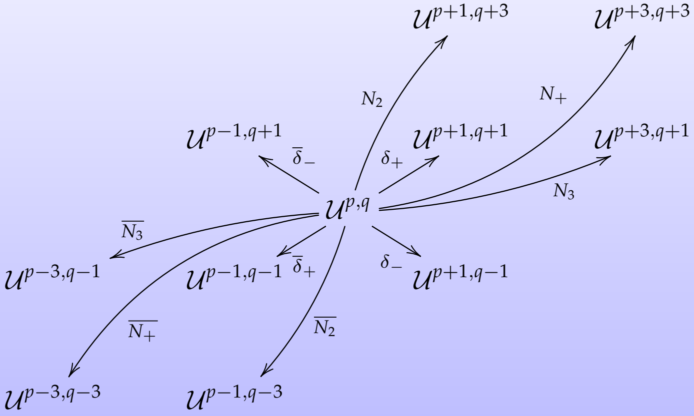
$$[\![\Gamma(V_-^{1,0}), \Gamma(V_-^{1,0})]\!] \subset \Gamma(V_-^{1,0}).$$

Nijenhuis tensor and intrinsic torsion



Components of d^H for a (positive) SKT structure.

Nijenhuis tensor and intrinsic torsion



Components of d^H for a (negative) SKT structure.

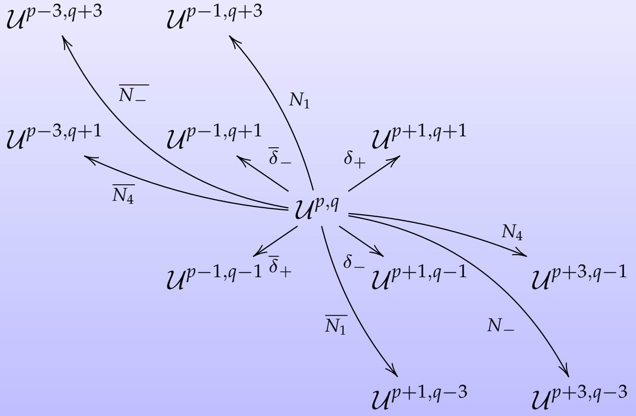
Nijenhuis tensor and intrinsic torsion

Graphic proof of Gualtieri's theorem

Theorem (Gualtieri 04)

A generalized Kähler structure is equivalent to a pair of positive and negative SKT structures.

Nijenhuis tensor and intrinsic torsion



Components of d^H for a (positive) SKT structure.

Nijenhuis tensor and intrinsic torsion

Let $W^k = \oplus_{p+q=k} U^{p,q}$ and $\mathcal{W}^k = \Gamma(W^k)$.

Proposition

A generalized almost Hermitian structure is an SKT structure if and only if

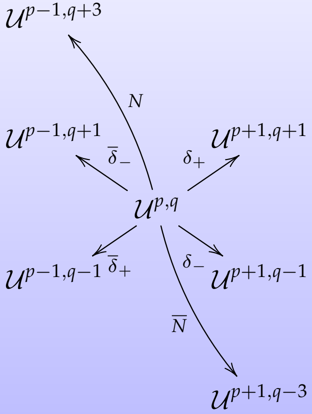
$$d^H : \mathcal{W}^k \longrightarrow \mathcal{W}^{k-2} \oplus \mathcal{W}^k \oplus \mathcal{W}^{k+2}.$$

$$\delta_+^N : \mathcal{W}^k \longrightarrow \mathcal{W}^{k+2};$$

$$\overline{\delta_+^N} : \mathcal{W}^k \longrightarrow \mathcal{W}^{k-2};$$

$$\not{\delta}_- : \mathcal{W}^k \longrightarrow \mathcal{W}^k.$$

Nijenhuis tensor and intrinsic torsion



Components of d^H for a gen. cplx. extension of an SKT structure.

Nijenhuis tensor and intrinsic torsion

Proposition

A generalized Hermitian structure is an SKT structure if and only if

$$d^H : \mathcal{U}^{p,q} \longrightarrow \mathcal{U}^{p-1,q+3} \oplus \mathcal{U}^{p-1,q+1} \oplus \mathcal{U}^{p-1,q-1} \oplus \mathcal{U}^{p+1,q+1} \oplus \mathcal{U}^{p+1,q-1} \oplus \mathcal{U}^{p+1,q-3}.$$

Hodge theory

In $(M^m, \mathcal{G}, or)$ we define

$$\not{D}_+ = \frac{1}{2}(d^H + (-1)^{m+1}(d^H)^*)$$

$$\not{D}_- = \frac{1}{2}(d^H + (-1)^{m}(d^H)^*)$$

Then:

$$\not{D}_+ : \Omega^{\bullet}_{\pm}(M) \longrightarrow \Omega^{\bullet}_{\mp}(M)$$
$$\not{D}_- : \Omega^{\bullet}_{\pm}(M) \longrightarrow \Omega^{\bullet}_{\pm}(M)$$

$$(-1)^{m+1}\not{D}_+^2 = (-1)^m \not{D}_-^2 = \tfrac{1}{4}\triangle_{d^H}.$$

Hodge theory

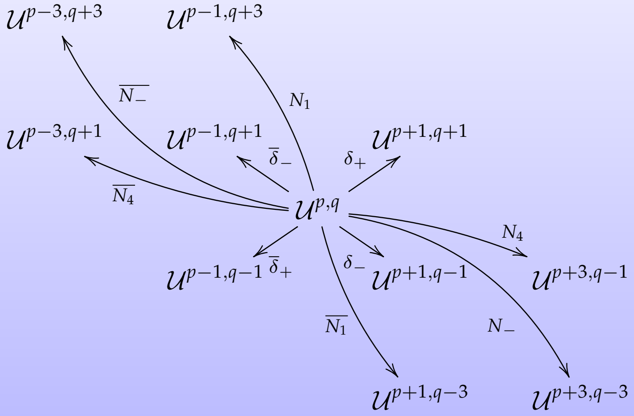
Theorem

In a compact SKT manifold, the d^H-cohomology splits according to the $\mathcal{W}^k$ decomposition of forms.

Remark: *The theorem also holds for parallel (almost) Hermitian structures with closed, skew torsion.*

Hodge theory

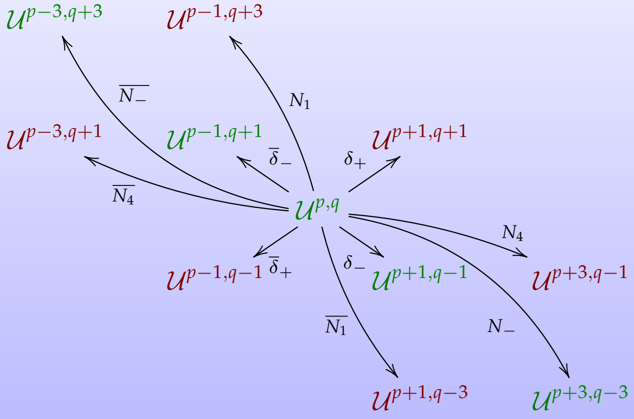
Proof:



Components of d^H for a (positive) SKT structure.

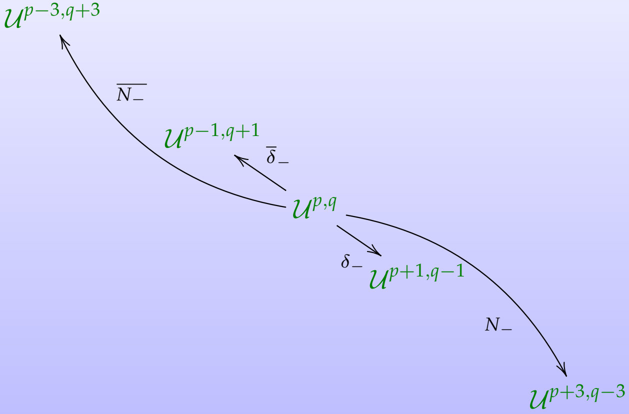
Hodge theory

Proof:



Components of d^H for a (positive) SKT structure.

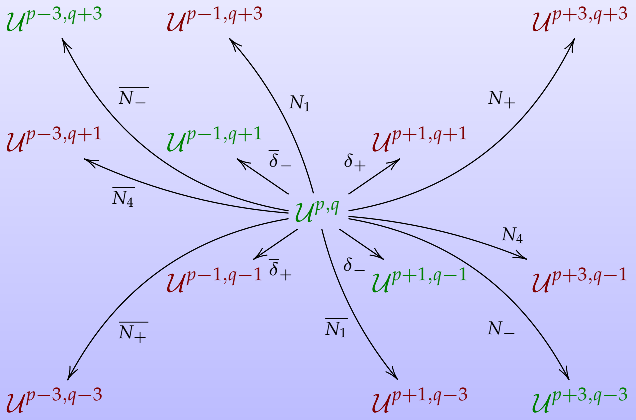
Hodge theory



Components of $\not{D}_-$ for a (positive) SKT structure.

Hodge theory

Proof: (Parallel case)



Components of d^H for a parallel positive Hermitian structure.

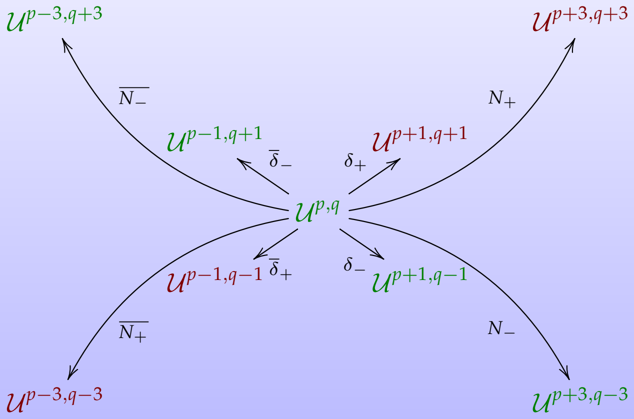
Hodge theory

Corollary (Parallel bi-Hermitian structures)

Let (M, g) be a compact Riemannian manifold and $H \in \Omega^3_{cl}(M)$. If the metric connections with torsion $\pm H$ have holonomy in $U(n)$ the d^H-cohomology splits according to the $\mathcal{U}^{p,q}$ decomposition of forms.

Hodge theory

Proof:



Components of d^H for a parallel bi-Hermitian structure.

Hodge theory

Theorem

Let (M, g) be a compact Riemannian manifold and $H \in \Omega^3_{cl}(M)$. If the metric connections with torsion $\pm H$ have holonomy in $G_\pm$ and $\mathfrak{g}_\pm \subset \wedge^2 V_\pm$ are the Lie algebras of $G_\pm$, the d^H-cohomology splits according to the irreducible representation of

$$\mathfrak{g}_+ \times \mathfrak{g}_- \subset \wedge^2 V_+ \times \wedge^2 V_- \subset \wedge^2 (T \oplus T^*).$$

Kähler vs. GK vs. SKT vs. reduced holonomy

	Kähler	GK	SKT	red. hol.
Decomposition of cohomology	✓	✓	✓	✓
Hodge theory	✓	✓	✓	✗
Frölicher spectral seq. deg.	✓	✓	✓	✗
Formality	✓	✗	✗	?
Unobstr. deformations	✗	✗	✗	?

Hodge theory

Theorem

In a compact SKT manifold we have

$$\triangle_{\delta_+^N} = \triangle_{\overline{\delta_+^N}} = \frac{1}{4}\triangle_{d^H}.$$

Proof:

Integration by parts & $\star|_{\mathcal{U}^{p,q}} = -i^{p+q}$ implies that $(\delta_+^N)^* = -\overline{\delta_+^N}$.

$$\not{D}_+ = \delta_+^N + \overline{\delta_+^N} = \delta_+^N - \delta_+^{N*}$$

$$\frac{1}{4}\triangle_{d^H} = -\not{D}_+^2 = -(\delta_+^N - \delta_+^{N*})^2 = \triangle_{\delta_+^N}.$$

Hodge theory

Theorem

In a compact SKT manifold (M, g, I), the $\partial + i\bar{\partial}\omega$ cohomology is isomorphic to the d^H-cohomology.

Proof: The automorphism of $\wedge^\bullet T^*_{\mathbb{C}}M$

$$\Psi : \Omega^\bullet(M;\mathbb{C}) \longrightarrow \Omega^\bullet(M;\mathbb{C}) \qquad \Psi(\varphi) = e^{i\omega} e^{\frac{i\omega^{-1}}{2}} \varphi$$

satisfies

$$\Psi\partial = \bar{\delta}_+ \qquad \text{and} \qquad \Psi(2i\bar{\partial}\omega) = N$$

Hodge theory

Corollary

In a compact SKT manifold (M, g, I), the spectral sequence correspoding to the decomposition

$$d^H = (\partial + i\overline{\partial}\omega) + (\overline{\partial} - i\partial\omega)$$

degenerates at the second page.

Deformations

- Deformations are given by the action of $\mathrm{SO}(T \oplus T^*)$.
- Small deformations are given by the action of (exponential of) elements in the Lie algebra
$$\Gamma(\mathfrak{spin}(T \oplus T^*)).$$
- It is natural to consider the question of deformations in the context of stability.

Deformations

Question

Which deformations of $\mathcal{J}_1$ can be completed with a deformation of $\mathcal{G}$ (or $\mathcal{J}_2$) so that $(\mathcal{G}, \mathcal{J}_1)$ is a positive SKT structure?

Deformations

Deformations of $\mathcal{J}_1$ are determined by

$$e^{\alpha}, \qquad \alpha \in \Gamma(\wedge^2 \overline{L_{\mathcal{J}_1}})$$

And lead to consider the operator

$$e^{-\alpha} d^H e^{\alpha} \overset{linearization}{\rightsquigarrow} \{d^H, \alpha\}.$$

with respect to the $\mathcal{U}^{p,q}$ splitting.
Here, the natural differential operators are

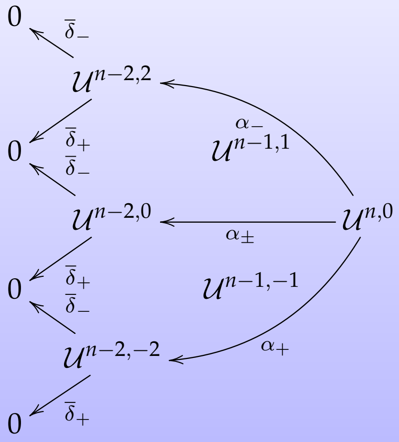
$$\partial_{\pm} = \{\delta_{\pm}, \cdot\};$$

$$\overline{\partial}_{\pm} = \{\overline{\delta}_{\pm}, \cdot\}$$

$$\mathcal{N} = \{N, \cdot\}$$

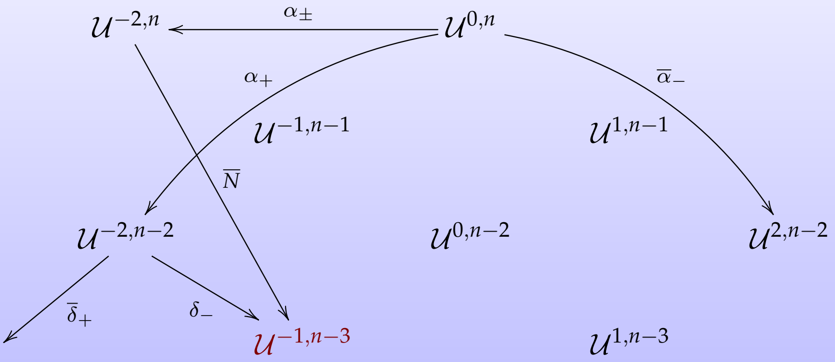
Deformations

Linear action of α is given by



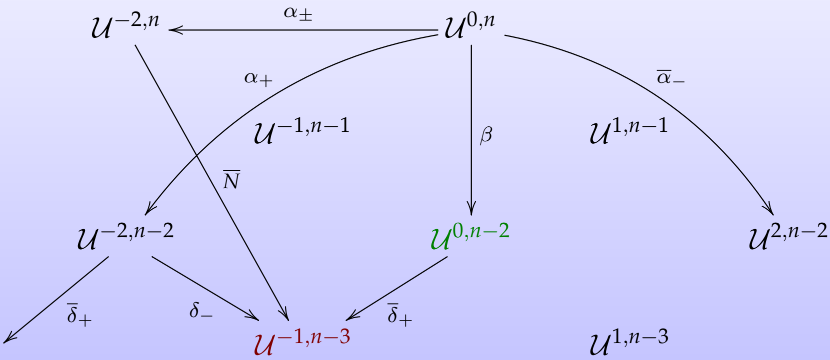
Deformations

For $\mathcal{J}_2$



Deformations

Can change α by an element $\beta \in \Gamma(V_+^{0,1} \otimes V_-^{1,0})$:



Need:

$$\overline{\partial}_+\beta = -(\partial_-\alpha_+ + \overline{\mathcal{N}}\alpha_\pm)$$

Deformations

Theorem

The obstructions to deforming an SKT structure lie in $H^{2,1}_{\overline{\partial}_+}(M)$. *If this space vanishes, any deformation of* $\mathcal{J}_1$ *can be completed to a deformation of the SKT structure.*

Theorem

If $\alpha = \alpha_- \in \Gamma(\wedge^2 V_-^{0,1})$, *then the deformed structure is still SKT*

Corollary

If (M, I, ω) *is Kähler, deformations of the symplectic form turn it into an SKT structure.*