GENERALIZED GEOMETRY OF TYPE B_n

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- generalized geometry on M^n

- $T \oplus T^*$

- inner product $(X+\xi, X+\xi) = i_X\xi$

- $SO(n,n)$-structure

- generalized geometry on M^n

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- inner product $(X+\xi, X+\xi) = i_X \xi$

- $SO(n,n)$-structure $-$ type D_n

- “extended” generalized geometry on M^n
- $T \oplus 1 \oplus T^*$
- inner product $(X+\lambda+\xi, X+\lambda+\xi) = i_X\xi + \lambda^2$
- $SO(n+1,n)$-structure $-$ type B_n

C M Hull, *Generalised geometry for M-theory,* JHEP **07** (2007)

D Baraglia, *Leibniz algebroids, twistings and exceptional generalized geometry*, arXiv: 1101.0856

P Bouwknegt, *Courant algebroids and generalizations of geometry*, String-Math 2011

Roberto Rubio

Mario Garcia Fernandez

- $\mathfrak{so}(n+1,n) = \Lambda^2 T + T + \operatorname{End} T + T^* + \Lambda^2 T^*$

- $A \in T^*, B \in \Lambda^2 T^*$

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- $X + \lambda + \xi \mapsto X + (\lambda + i_X A) + (\xi - 2\lambda A - i_X AA)$

- $X + \lambda + \xi \mapsto X + \lambda + \xi + i_X B$

ACTION ON VECTORS

- $\mathfrak{so}(n+1,n) = \Lambda^2 T + T + \mathrm{End}\, T + T^* + \Lambda^2 T^*$
- $A \in T^*, B \in \Lambda^2 T^*$
- $X + \lambda + \xi \mapsto X + (\lambda + i_X A) + (\xi - 2\lambda A - i_X AA)$
- $X + \lambda + \xi \mapsto X + \lambda + \xi + i_X B$
- $(A, B).(A', B') = (A + A', \underline{B + B' - 2A \wedge A'})$

SPINORS

- exterior algebra $\Lambda^* T^*$

- $\tau(\varphi) = (-1)^{\deg \varphi} \varphi$

- Clifford multiplication $(X + \lambda + \xi) \cdot \varphi = i_X \varphi + \lambda \tau \varphi + \xi \wedge \varphi$

- $\exp B(\varphi) = e^{-B} \wedge \varphi$

- $\exp A(\varphi) = e^{-A\tau} \varphi = \varphi - A \wedge \tau \varphi$

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AUTOMORPHISMS

- generalized diffeomorphisms $\mathsf{GDiff}(M) = \mathsf{Diff}(M) \rtimes \Omega_{cl}^{2+1}$
- $\Omega_{cl}^{2+1} = \{(A, B) \in \Omega^1 \times \Omega^2 : dA = dB = 0\}$
- $(A, B).(A', B') = (A + A', B + B' - 2A \wedge A')$
- $0 \to \Omega_{cl}^2 \to \Omega_{cl}^{2+1} \to \Omega_{cl}^1 \to 0$

central extension

- generalized vector field $u = X + \lambda + \xi$
- $u \mapsto X - d\lambda - d\xi \in \mathfrak{gdiff}(M)$
- Lie derivative on $T + 1 + T^*$ = Dorfman "bracket"
- Lie derivative on spinors $d(u \cdot \varphi) + u \cdot d\varphi$

- skew-symmetrize action on $T+1+T^*$, get a Courant bracket

- $[u, fv] = f[u, v] + (Xf)v - (u, v)df$

- $$[[u, v], w] + \ldots = \frac{1}{3}d(([u, v], w) + \ldots)$$

Z Chen, M Stienon and P Xu, *On regular Courant algebroids*, arXiv: 0909.0319

WHAT'S NEW?

- $\mathbf{G}_2^2$ -STRUCTURES

- TWISTING AND COHOMOLOGY

- CONNECTIONS, TORSION AND CURVATURE

G_2^2 MANIFOLDS

- $G_2^2 \subset SO(4,3)$
- stabilizer of a non-null spinor
- $\rho \in \Lambda^* T^*$, $\rho = \rho_0 + \rho_1 + \rho_2 + \rho_3$
- $\langle \rho, \rho \rangle = \rho_0 \rho_3 - \rho_1 \wedge \rho_2 \neq 0$

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- $\langle \rho, \rho \rangle = \rho_0 \rho_3 - \rho_1 \wedge \rho_2 \neq 0$
- G_2^2 manifold: M^3 + closed form ρ with $\langle \rho, \rho \rangle \neq 0$

$$\{f, g\} = \frac{df \wedge dg \wedge \rho_1}{\langle \rho, \rho \rangle}$$

Poisson structure

EXAMPLES

- $\rho = 1 + \rho_3$, ρ_3 volume form.

- $\rho_0 = 0 \Rightarrow \rho_1 \wedge \rho_2 \neq 0$

- non-vanishing closed 1-form $\rho_1 \Rightarrow M^3$ fibres over a circle.

- $p : M \to S^1$, $|\rho_1 - \lambda p^* d\theta| < \epsilon$, $\lambda \in \mathbf{Q}$

- $\rho = \rho_1 + \rho_2$

- unique vector field X such that $i_X \rho_2 = 0, i_X \rho_1 = 1$

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- unique vector field X such that $i_X \rho_2 = 0, i_X \rho_1 = 1$

- $p : M \to S^1$, $p^* d\theta = \rho_1$

- $\rho_2 =$ symplectic form on fibre, $\mathcal{L}_X \rho_2 = d(i_X \rho_2) = 0$

- $\Rightarrow$ mapping torus of a symplectic diffeomorphism of surface

QUESTIONS

- IF ρ, ρ' ARE COHOMOLOGOUS AND SUFFICIENTLY CLOSE, ARE THEY EQUIVALENT UNDER A GENERALIZED DIFFEOMORPHISM?

- WHICH COHOMOLOGY CLASSES CONTAIN G_2^2 STRUCTURES?

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- $\rho' - \rho = d\varphi$

- $\rho_t = \rho + t d\varphi$

- find generalized vector field u_t such that $u_t \cdot \rho_t = -\varphi$

- ..... integrate to a generalized diffeomorphism

(Moser argument)

- $u \in T + 1 + T^*, \rho \in \Lambda^* T^*$

- $\pi : \Lambda^* T^* \to \Lambda^* T^* / \Lambda^3 T^*$

- $\rho_0 \neq 0$ then $u \mapsto \pi(u \cdot \rho)$ is invertible

- $\rho_0 = 0$? Assume $\rho_1 = p^* d\theta$

- image $u \mapsto u \cdot \rho$ is $\{\psi : \langle \rho, \psi \rangle = 0\}$

- $\varphi_2 \mapsto \varphi_2 + \lambda \rho_2 \Rightarrow$

  $\langle \rho, \varphi \rangle = -\rho_1 \wedge \varphi_2 + \rho_2 \wedge \varphi_1 - \rho_3 \varphi_0 = d\alpha$

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- integrate over fibres, $\alpha - f\rho_2 = d\beta + \rho_1 \wedge \gamma$

- $d\alpha = df \wedge \rho_2 - \rho_1 \wedge d\gamma$

QUESTIONS

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- WHICH COHOMOLOGY CLASSES CONTAIN G_2^2 STRUCTURES?

- $\rho_0 \neq 0$, use $A = \rho_1/\rho_0, B = \rho_2/\rho_0$
- transform to $\rho_0 + (\rho_3 - \rho_1 \wedge \rho_2/\rho_0)$
- $\Rightarrow \rho$ equivalent to $\rho_0(1 + vol)$

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- $\underline{\rho_0 = 0}$: which classes in $H^1(M^3, \mathbf{R})$ are represented by non-vanishing 1-forms?

- Answer: cone on faces of the unit ball in the Thurston norm.

TWISTING

- D_n generalized geometry, $T+T^*$

- identify $T+T^*|_U$ with $T+T^*|_V$ over $U\cap V$

- $X+\xi\mapsto X+\xi+i_X B_{UV}$

- B_{UV} 1-cocycle in Ω^2_{cl}

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- $X + \xi \mapsto X + \xi + i_X B_{UV}$
- B_{UV} 1-cocycle in Ω^2_{cl}
- Courant algebroid $0 \to T^* \to E \to T \to 0$

- B_n generalized geometry, $T + 1 + T^*$
- identify with (A_{UV}, B_{UV}) 1-cocycle in Ω^{2+1}_{cl}
- Courant algebroid $0 \to T^* \to E \to L \to 0$
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- B_n generalized geometry, $T + 1 + T^*$

- identify with (A_{UV}, B_{UV}) 1-cocycle in Ω_{cl}^{2+1}

- Courant algebroid $0 \to T^* \to E \to L \to 0$

- $0 \to 1 \to L \to T \to 0$

- L Lie algebroid (like $TP/U(1)$ for P a $U(1)$-bundle over M)

- $[(A_{UV}, B_{UV})] \in H^1(M, \underline{\Omega}_{cl}^{2+1})$

- $0 \to \underline{\Omega}_{cl}^2 \to \underline{\Omega}_{cl}^{2+1} \to \underline{\Omega}_{cl}^1 \to 0$ sheaves

- $H^1(M, \underline{\Omega}_{cl}^2) \to H^1(M, \underline{\Omega}_{cl}^{2+1}) \to H^1(M, \underline{\Omega}_{cl}^1) \xrightarrow{\delta} H^2(M, \underline{\Omega}_{cl}^2)$

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- $$\begin{array}{ccccccc} H^1(M, \underline{\Omega}_{cl}^2) & \to & H^1(M, \underline{\Omega}_{cl}^{2+1}) & \to & H^1(M, \underline{\Omega}_{cl}^1) & \xrightarrow{\delta} & H^2(M, \underline{\Omega}_{cl}^2) \\ \| & & & & \| & & \| \\ H^3(M, \mathbf{R}) & & & & H^2(M, \mathbf{R}) & & H^4(M, \mathbf{R}) \end{array}$$

- $\delta(x) = x^2$

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- $\delta(x) = x^2$

- $F \in \Omega_{cl}^2 \quad F^2 = -dH$

- $\Lambda^* T^*|_U, \Lambda^* T^*|_V$, identify with $\exp(-A_{UV}\tau + B_{UV})$

- spinor bundle S, differential d

- $\Lambda^* T^*|_U, \Lambda^* T^*|_V$, identify with $\exp(-A_{UV}\tau + B_{UV})$
- spinor bundle S, differential d
- or... split extension, and get
- $d + F\tau + H : \Omega^* \to \Omega^*$
- $(d + F\tau + H)^2 = 0$

WHAT IS THIS COHOMOLOGY?

- Suppose $F =$ curvature of a $U(1)$-connection
- connection form θ on principal bundle P
- $d(H + \theta F) = 0$
- $\Rightarrow P$ is T-dual to itself

- $T(\alpha + \theta\beta) = \beta - \theta\alpha$

- $T^{-1}(d + H + \theta F)T = -(d + H + \theta F)$

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- $T^{-1}(d + H + \theta F)T = -(d + H + \theta F)$
- $(d + H + \theta F)(\alpha + \theta\tau\alpha) = (1 + \theta\tau)(d\alpha + F\tau\alpha + H\alpha)$
- $\tau T(\alpha + \theta\tau\alpha) = \tau(\tau\alpha - \theta\alpha) = \alpha + \theta\tau\alpha$

- $T\tau = -\tau T \Rightarrow (\tau T)^2 = 1$

- $\bar{H} = H + \theta F$ closed 3-form

- (H, F)-twisted cohomology of $M \cong$

  τT-invariant part of $d + \bar{H}$-cohomology of P.

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  τT-invariant part of $d + \bar{H}$-cohomology of P.
- Generalization of twisted K-theory?

CONNECTIONS AND TORSION

CONNECTIONS

- generalized connection on $W = D : W \to E \otimes W$
- $D(fs) = fDs + df \otimes s \ (df \in T^* \subset E)$
- $=$ ordinary connection $+$ section of $E \otimes \operatorname{End} W$
- $u \in E, D_u s = (Ds, u) \in W$
- frame bundle, principal bundle...

TORSION

- generalized connection $D : E \to E \otimes E$

- torsion for affine connection ∇: $\nabla_X Y - \nabla_Y X - [X, Y]$

- Courant bracket $[u, fv] = f[u, v] + (Xf)v - (u, v)df$

- $T(u, v, w) = (D_u v - D_v u - [u, v], w) + \frac{1}{2}(D_w u, v) - \frac{1}{2}(D_w v, u)$

TORSION

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D preserves inner product $\Rightarrow$ torsion in $\Lambda^3 E$

- $T^* \xrightarrow{\nabla} T^* \otimes T^* \xrightarrow{\pi} \Lambda^2 T^*$

- torsion for affine connection $= p \circ \nabla - d$

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- spinors S, $D : S \to E \otimes S$

- "Dirac" operator: $S \xrightarrow{D} E \otimes S \xrightarrow{\text{Cliff}} S$

- torsion $= \text{Cliff} \circ D - d$

METRICS

- generalized metric: reduction of $SO(n+1,n)$ structure of $T+1+T^*$ (or E) to $S(O(n+1)\times O(n))$
- $\Leftrightarrow$ rank n subbundle $V\subset E$ on which induced metric is negative definite.
- $\Rightarrow V^\perp$ positive definite
- e.g. $V=\{X-gX\in T+1+T^*\}, V^\perp=\{X+\lambda+gX\}$

PROBLEM:

FIND A TORSION-FREE GENERALIZED CONNECTION WHICH PRESERVES THE GENERALIZED METRIC

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- $E = V \oplus V^{\perp}$

- $V \cong T, V^{\perp} \cong 1 \oplus T$

- take Levi-Civita ∇ and trivial connection on 1 and add on terms so that Cliff $\circ\, D = d$

A Coimbra, C Strickland-Constable and D Waldram, *Supergravity as generalized geometry I: Type II theories*, arXiv: 1107.1733

- $V_- = \{X^- = X - gX\}, V_+ = \{X^+ = X + gX\}, V_0 = 1$
- add terms in $\Lambda^2 V_- \otimes (V_- + V_0 + V_+)$ and

  $\Lambda^2(V_0 + V_+) \otimes (V_- + V_0 + V_+)$
- so that $\mathrm{Cliff} \circ D = d + F\tau + H$

- $$d\varphi = gX_i \wedge \nabla_i \varphi = \frac{1}{2}(X_i^+ - X_i^-) \cdot \nabla_i \varphi$$

- $$H \wedge \varphi = \frac{1}{48} H_{ijk}(X_i^+ - X_i^-) \cdot (X_j^+ - X_j^-) \cdot (X_k^+ - X_k^-) \cdot \varphi$$

- $$F \wedge \tau\varphi = \frac{1}{8} F_{ij}(X_i^+ - X_i^-) \cdot (X_j^+ - X_j^-) \cdot X_0 \cdot \varphi$$

- $\Lambda^2 V_- \otimes (V_- + V_0 + V_+)$

- $(H \wedge \varphi)^{---} = -\frac{1}{48} H_{ijk} X_i^- \cdot X_j^- \cdot X_k^- \cdot \varphi$

  $(H \wedge \varphi)^{--+} = \frac{1}{16} H_{ijk} X_i^- \cdot X_j^- \cdot X_k^+ \cdot \varphi$

- $(F \wedge \tau\varphi)^{--0} = \frac{1}{8} F_{ij} X_i^- \cdot X_j^- \cdot X_0 \cdot \varphi$

CURVATURE

- $D_u D_v - D_v D_u - D_{[u,v]}$?

- Courant bracket $[u, fv] = f[u,v] + (Xf)v - (u,v)df$

- OK if $(u,v) = 0$

- Curvature $\in \mathrm{End}\, E \otimes V \otimes V^{\perp}$

RICCI CONTRACTIONS

- $(\mathsf{End}\, V \oplus \mathsf{End}\, V^{\perp}) \otimes V \otimes V^{\perp}$

- $\mathsf{End}\, V \otimes V \otimes V^{\perp} \mapsto V \otimes V^{\perp}$

  $\mathsf{End}\, V^{\perp} \otimes V \otimes V^{\perp} \mapsto V \otimes V^{\perp}$

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  $\mathsf{End}\, V^{\perp} \otimes V \otimes V^{\perp} \mapsto V \otimes V^{\perp}$

- generalized metric $V \subset E$

- tangent vector $\in \mathsf{Hom}(V, V^{\perp})$

- define generalized Ricci flow

RICCI CURVATURES

- $\mathrm{End}\, V \otimes V \otimes V^{\perp} \mapsto V \otimes V^{\perp}$

- $V \otimes V^{\perp} \cong T \otimes (1 + T) \cong \mathrm{Sym}^2\, T^* + \Lambda^2 T^* + T^*$

$$\mathrm{Sym}^2\, T^* : R_{i\ell} - \frac{1}{4} H_{ijk} H_{\ell}^{jk}$$

$$\Lambda^2 T^* : \nabla^i H_{ijk}$$

RICCI CURVATURES

- End $V^{\perp} \otimes V \otimes V^{\perp} \mapsto V \otimes V^{\perp}$

$$\boxed{\begin{aligned} \mathrm{Sym}^2\, T^* &: R_{i\ell} - \frac{1}{4} H_{ijk} H_{\ell}^{jk} - F_{ij} F^{j\ell} \\ \Lambda^2 T^* &: -\nabla^i H_{ijk} \\ T^* &: \nabla^i F_{ij} + \frac{1}{2} F^{ki} H_{jki} \end{aligned}}$$

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heterotic supergravity $+$ $U(1)$ gauge field