Grundlehren der mathematischen Wissenschaften
A Series of Comprehensive Studies in Mathematics

Camille Laurent-Gengoux · Anne Pichereau · Pol Vanhaecke

Poisson Structures

Poisson structures appear in a large variety of contexts, ranging from string theory, classical/quantum mechanics and differential geometry to abstract algebra, algebraic geometry and representation theory. In each one of these contexts, it turns out that the Poisson structure is not a theoretical artifact, but a key element which, unsolicited, comes along with the problem that is investigated, and its delicate properties are decisive for the solution to the problem in nearly all cases. Poisson Structures is the first book that offers a comprehensive introduction to the theory, as well as an overview of the different aspects of Poisson structures. The first part covers solid foundations, the central part consists of a detailed exposition of the different known types of Poisson structures and of the (usually mathematical) contexts in which they appear, and the final part is devoted to the two main applications of Poisson structures (integrable systems and deformation quantization).

The clear structure of the book makes it adequate for readers who come across Poisson structures in their research or for graduate students or advanced researchers who are interested in an introduction to the many facets and applications of Poisson structures.

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Springer

Intersection cohomology of coisotropic submanifolds

Work in progress

Gerstenhaber algebra, Gerstenhaber module.

A *Gerstenhaber algebra* is a graded vector space $\mathcal{E}$, endowed with

1. a graded commutative product $\wedge$ on $\mathcal{E}$
2. a graded Lie algebra structure $[.,.]$ on $\mathcal{E}[-1]$,
3. + compatibility: $[Q \wedge R, P] = [Q, P] \wedge R + (-1)^{q(p-1)} Q \wedge [R, P]$ for all P, Q, R of degrees p, q, r.

Relaxing Jacobi for $[.,.]$ leads to *pre-Gerstenhaber algebra*.

A *Gerstenhaber algebra module* is a graded vector space $\mathcal{F}$ endowed with:

1. a structure $\imath$ of module for the graded algebra $(\mathcal{E}, \wedge)$,
2. a structure $\mathcal{L}$ of module for the graded Lie algebra $(\mathcal{E}[-1], [.,.])$),
3. + compatibility: $\mathcal{L}_P \circ \imath_Q \ \alpha - (-1)^{q(p-1)} \imath_Q \circ \mathcal{L}_P \ \alpha = \imath_{[P,Q]} \ \alpha$.

Corresponding notion of a *pre-Gerstenhaber module*.

(Pre)-Lie algebroid

Definition

Let $A \to M$ be a vector bundle. A *pre-Lie algebroid structure* on A is a graded derivation of degree $+1$:

$$D : \Gamma(\wedge^{\bullet} A^*) \to \Gamma(\wedge^{\bullet+1} A^*),$$

said to be a *Lie algebroid* when it squares to 0.

The *bracket* of $P \in \Gamma(\wedge^p A)$ with $Q \in \Gamma(\wedge^q A)$ is the unique element $R := [P, Q]_D \in \Gamma(\wedge^{p+q-1} A)$ s.t.

$$[\mathcal{L}_P, \imath_Q] = \imath_R.$$

In the previous, $\mathcal{L}_P := \imath_P \circ D - (-1)^p D \circ \imath_P$ is the *Lie derivative*.

Gerstenhaber algebra and Gerstenhaber module.

Proposition

Let $A \to M$ be a vector bundle. For every Lie algebroid structure D, the triple $(\Gamma(\wedge^\bullet A), \wedge, [\cdot,\cdot]_D)$ is a Gerstenhaber algebra, and $\Gamma(\wedge^\bullet A^*)$, equipped with:

1. the action of $(\Gamma(\wedge^\bullet A), \wedge)$ by contractions,
2. the action of $(\Gamma(\wedge^\bullet A), [.,.])$ by Lie derivatives,

is a module over this Gerstenhaber algebra.

This can be weakened as follows.

Gerstenhaber algebra and Gerstenhaber module.

Proposition

Let $A \to M$ be a vector bundle. For every **pre**-Lie algebroid structure D, the triple $(\Gamma(\wedge^\bullet A), \wedge, [\cdot,\cdot]_D)$ is a **pre**-Gerstenhaber algebra, and $\Gamma(\wedge^\bullet A^*)$, equipped with:

1. the action of $(\Gamma(\wedge^\bullet A), \wedge)$ by contractions,
2. the action of $(\Gamma(\wedge^\bullet A), [.,.]_D)$ by Lie derivatives,

is a module over this **pre**-Gerstenhaber algebra.

Koszul complex

Let $A \to M$ be a vector bundle. Choose a section $\varphi \in \Gamma(A^*)$.

1. Let $\imath_\varphi : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet-1} A)$ be the contraction by φ
2. Let $m_\varphi : \Gamma(\wedge^\bullet A^*) \to \Gamma(\wedge^{\bullet+1} A^*)$ be the (left) mutiplication by φ.

Both $\imath_\varphi$ and m_φ square to zero, hence define a homology $H_*(\varphi)$ and a cohomology $H^*(\varphi)$, said to be *attached to* φ. The graded algebra structure $\wedge$ and module structure $\imath$ go down:

Proposition

1. The homology $H_*(\varphi)$ is a graded commutative algebra.
2. The cohomology $H^*(\varphi)$ is a module over the algebra $H_*(\varphi)$.

Gerstenhaber structure on the Koszul complex.

Question 1. Let $A \to M$ be a vector bundle. Given

- a section $\varphi \in \Gamma(A^*)$,
- a Lie algebroid $D : \Gamma(\wedge^\bullet A^*) \mapsto \Gamma(\wedge^{\bullet+1} A^*)$,

do the Gerstenhaber algebra $(\Gamma(\wedge^\bullet A), \wedge, [.,.]_D)$, and its module $(\Gamma(\wedge^\bullet A^*), \imath, \mathcal{L})$, go to the quotient with respect to $m_\varphi, \imath_\varphi$ and define structures of Gerstenhaber algebra and module on $H_*(\varphi)$ and $H^*(\varphi)$ respectively ?

Answer: Yes, if $D(\varphi) = 0$.

Proposition

Let $A \to M$ be a vector bundle. For every Lie algebroid D and 1-cocyle $\varphi \in \Gamma(A^*)$, $H_*(\varphi)$ and $H^*(\varphi)$ are Gerstenhaber algebras and modules respectively.

Gerstenhaber structure on the Koszul complex.

Question 1. Let $A \to M$ be a vector bundle. Given

- a section $\varphi \in \Gamma(A^*)$,
- a **pre**-Lie algebroid $D : \Gamma(\wedge^\bullet A^*) \mapsto \Gamma(\wedge^{\bullet+1} A^*)$,

do the **pre**-Gerstenhaber algebra $(\Gamma(\wedge^\bullet A), \wedge, [.,.]_D)$, and its module $(\Gamma(\wedge^\bullet A^*), \imath, \mathcal{L})$, go to the quotient with respect to $m_\varphi, \imath_\varphi$ and define structures of **pre**-Gerstenhaber algebra and module on $H_*(\varphi)$ and $H^*(\varphi)$ respectively ?

Answer: Yes, if and only if $D(\varphi) = 0$.

Proposition

Let $A \to M$ be a vector bundle. For every **pre**-Lie algebroid D and 1-cocyle $\varphi \in \Gamma(A^*)$, $H_*(\varphi)$ and $H^*(\varphi)$ are **pre**-Gerstenhaber algebras and modules respectively.

Induced structures on Koszul complex-2.

Question 2. Could it be that D be only a **pre**-Lie algebroid but still the induced structure is a (honnest) Gerstenhaber algebra ?

Answer:

Theorem

Let $A \to M$ be a vector bundle. Given:

1. a section $\varphi \in \Gamma(A^*)$
2. a **pre**-Lie algebroid $D : \Gamma(\wedge^\bullet A^*) \to \Gamma(\wedge^{\bullet+1} A^*)$,

such that

- ♣ $D(\varphi) = 0$,
- ♠ $D^2 = C \circ m_\varphi + m_\varphi \circ C$ for some operator C (i.e. D^2 homotopic to zero - recall that $m_\varphi(\alpha) = \varphi \wedge \alpha$ computes $H^*(\varphi)$),

then D induces a structure of Gerstenhaber algebra on the homology $H_*(\varphi)$ and a structure of Gerstenhaber module on the cohomology $H^*(\varphi)$.

P_∞-algebras

Question 3. Where do pre-Lie algebroid structures D and sections φ satisfying ♣ (i.e. $D(\varphi) = 0$) and ♠ (i.e. $D^2 = C \circ m_\varphi + m_\varphi \circ C$) can arise from?

Answer. From a Maurer-Cartan element in a P_∞ algebras on $\wedge A^*$.

Definition

Let $A \to M$ be a vector bundle. A P_∞-algebra structure on $\wedge A^*$ a sequence of n-ary "brackets" :

$$[\Gamma(\wedge^{a_1} A^*), \ldots, \Gamma(\wedge^{a_n} A^*)]_n \subset \Gamma(\wedge^{a_1+\cdots+a_n-n+2} A^*)$$

for all $n \in \mathbb{N}^*$, such that

1. $[\cdots]_n$ skew-symmetric and is a derivation in each variable,
2. the "higher" Jacobi identities hold:

$$\sum_{i+j=n+1} \sum_{\sigma \in \Sigma_{i,j}} (-1)^{\sigma,x} [[x_{\sigma(1)}, \ldots, x_{\sigma(i)}]_i, x_{\sigma(i+1)}, \ldots, x_{\sigma(n+1)}]_j = 0.$$

Precisions on the degrees.

By construction:

$$[\Gamma(A^*), \Gamma(A^*)]_2 \subset \Gamma(\wedge^2 A^*)$$

(so sections of A^* do not come equipped with a bracket !)

Remark. For every $\varphi \in \Gamma(A^*)$ and every sequence $(a_n)_{n\geq 1}$ in $\mathbb{R}$, the operator

$$D : \alpha \mapsto \sum_{n\geq 1} a_n[\varphi, \ldots, \varphi, \alpha]_n$$

is a pre-Lie algebroid, provided that it converges.

Let $A \to M$ be a vector bundle, and $([\dots]_n)_{n\in\mathbb{N}^*}$ be a P_∞-structure on $\wedge A^*$.

Definition

A section $\varphi \in \Gamma(A^*)$ is said to be Maurer-Cartan when:

$$\sum_{n\in\mathbb{N}^*} \frac{[\varphi,\dots,\varphi]_n}{n!} = 0.$$

By construction:

$$D_\varphi(\alpha) := \sum_{n\in\mathbb{N}^*} \frac{[\varphi,\dots,\varphi,\alpha]_n}{n!}$$

is a pre-Lie algebroid that satisfies ♣ (i.e. $D_\varphi(\varphi)$).

From P_∞ algebra to pre-Gerstehanber algebra

Theorem

Let $A \to M$ be a vector bundle, equipped with a P_∞ structure on $\wedge A^*$.
Let $\varphi \in \Gamma(A^*)$ be a Maurer-Cartan element, then:

1. the operator
$$D_\varphi(\alpha) := \sum_{n \geq 1} \frac{1}{n!} [\varphi, \ldots, \varphi, \alpha]_n$$
is a pre-Lie algebroid (i.e. derivation of degree $+1$)
2. it satisfies the condition ♣, i.e $D_\varphi(\varphi) = 0$,
3. If, moreover, the P_∞-structure is quantizable by deformation in an A_∞-structure, then the condition ♠, i.e. $D_\varphi^2 = C \circ m_\varphi + m_\varphi \circ C$, is also satisfied.

Recall that $m_\varphi : \Gamma(\wedge^\bullet A^*) \to \Gamma(\wedge^{\bullet+1} A^*)$ is left mutiplication by φ.

Important: no need to quantize the Maurer-Cartan element.

From P_∞ algebra to pre-Gerstehanber algebra-2

Corollary

Given:

1. *a vector bundle $A \to M$,*
2. *a P_∞ structure on $\wedge A^*$,*
3. *a Maurer-Cartan element $\varphi \in \Gamma(A^*)$,*

then if:

1. *the P_∞ structure is quantizable by deformation,*
2. *all series considered converge,*

then the operator D_φ above induces a Gerstenhaber algebra structure on $H_*(\varphi)$ and a Gerstenhaber module structure on $H^*(\varphi)$.

Coisotropic submanifolds

Let (X, π) be a Poisson manifold. A submanifold $M \subset X$ is said to be *coisotropic* if one of the equivalent conditions is satisfied:

1. $\pi^{\#}(T_m M^{\perp}) \subset T_m M$
2. the ideal of functions vanishing on M is closed under Poisson bracket,
3. in a local adapted system of coordinates $(x_1, \ldots, x_n, p_1, \ldots, p_d)$, the Poisson structure is of the form

$$\pi = \sum_{i,j} a_{i,j} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j} + \sum_{i,k} b_{i,k} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial x_k} + \sum_{k,l} c_{k,l} \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial x_l},$$

   where the functions $a_{i,j}$ vanish identically on M.

Facts:

1. (X, π) Poisson $\Rightarrow$ T^*X Lie algebroid, $\Rightarrow$ the operator $[\pi, \cdot]$ is a derivation squaring to 0 of $\Gamma(\wedge^\bullet TX)$.
2. $M \subset X$ coisotropic $\Rightarrow$ $TM^{\perp}$ Lie sub-algebroid.

P_∞-algebras and coisotropic submanifolds

Important example [Oh-Park]. Given

1. a Poisson manifold (X, π),
2. a coisotropic submanifold M,
3. a global transverse linear structure,

there is an induced P_∞ structure on $\wedge(T_M X / TM)$, constructed as follows:

$$[P_1, \ldots, P_n]_n := p(\left[\left[\pi, \widehat{P_1}\right], \ldots, \widehat{P_n}\right])$$

where $\widehat{P_1}, \ldots, \widehat{P_n}$ are the unique multivector vector fields invariant by translation along the fibers that extend $P_1, \ldots, P_n$, and p is the operator that projects on $\Gamma(\wedge^\bullet T_M X / TM)$ a multivector field on X.

Moreover, [Cattaneo-Felder] this P_∞-structure admits a quantization by deformation.

Maurer-Cartan and coisotropic submanifolds.

Maurer-Cartan elements encodes (formal) deformations of coisotropic submanifolds.

Example: Given two submanifolds M, N of the same dimension of a algebraic Poisson manifold, there exists, in a neighborhood of every point in M, *adapted coordinates* $(x_1, \ldots, x_n, p_1, \ldots, p_d)$ for M such that:

1. M is given by $p_1 = \cdots = p_d = 0$,
2. N is given by $p_1 - \varphi_1(x_1, \ldots, x_n) = \cdots = p_d - \varphi_d(x_1, \ldots, x_n) = 0$.

Assume M is coisotropic. Then

$$\varphi := \sum_i \varphi_i \frac{\partial}{\partial p_i}$$

is a Maurer-Cartan element w.r.t. the P_∞-structure (when seen as a section of $\Gamma(T_M X / TM)$) iff N coisotropic [Schätz-Zambon].

Conclusion 1

General idea:

1. X Poisson + M coisotropic $\rightsquigarrow$ P_∞-structure on $\Gamma(\wedge^\bullet T_M X/TM)$.
2. N coisotropic $\rightsquigarrow$ Maurer-Cartan element $\varphi \in \Gamma(T_M X/TM)$.
3. Previous theorem $\rightsquigarrow$ Gerstenhaber algebra structure on $H_*(\varphi)$ and Gerstenhaber module on $H^*(\varphi)$.

More precisely:

Corollary

Let (X, π) be an algebraic Poisson manifold. Let M, N be a coisotropic submanifolds of the same dimension. Choose a system of adapted coordinates in a neighborhood of a point in $M \cap N$. Construct φ as above. Then the homologies $H_*(\varphi)$ and co-homologies $H^*(\varphi)$ attached to φ admits induced Gerstenhaber algebra structures and modules respectively.

Gluing of the previous homologies and structures.

Baranovski-Ginzburg (following Behrend-Fantechi) have constructed a Gerstenhaber algebra structure on the sheafified $Tor_X(M, N)$ and $Ext_X(M, N)$ of coisotropic submanifolds of an algebraic Poisson manifold.

Proposition

Let M, N be submanifolds of the same dimension in X. Let $U \subset X$ be an open subset on which there exists adapted coordinates, and let $\varphi = \sum_i \varphi_i \frac{\partial}{\partial p_i}$ be as before. Then $Tor(M \cap U, N \cap U) = H_*(\varphi)$ and $Ext(M \cap U, N \cap U) = H^*(\varphi)$.

Question. Does our construction match [BG] ?

Yes in the symplectic case (direct computation).

Yes in a Poisson case, due to the existence of symplectic realization ?